Arithmetic Quantum Chaos – An Introduction

Lior Silberman[†]

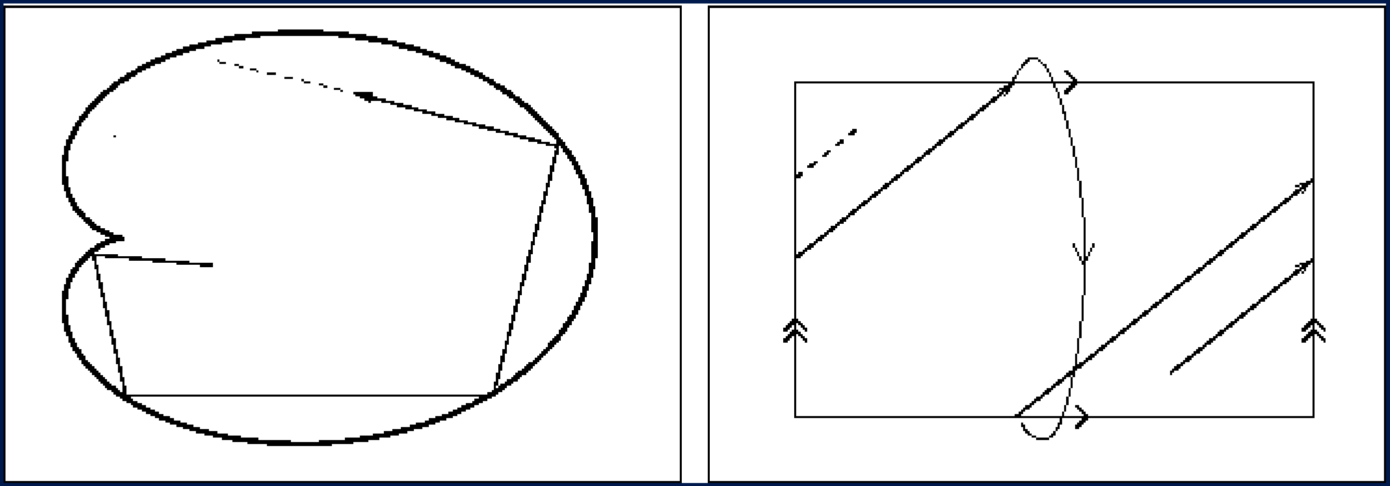
December 1st, 2005

[†]School of Mathematics, Institute for Advanced Study

Homepage: http://www.math.ias.edu/~lior

Mechanics of a Free Particle

Classical Mechanics - Examples



Ex. 1: (Cardioid) planar domain with a piecewise smooth boundary.

Ex. 2: (Flat Torus) compact Riemannian manifold (M^n, ds^2).

Ex. 3: Surfaces of constant negative curvature

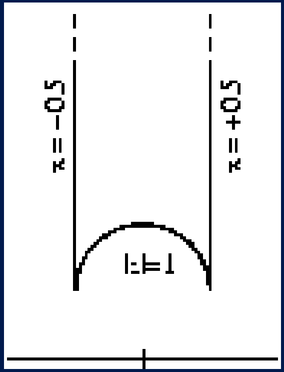
- $\mathbb{H} = \{x + iy \mid y > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$.
- $G = \mathrm{SL}_2(\mathbb{R})$ acts by the isometries $z \mapsto \frac{az+b}{cz+d}$.
- $K = \mathrm{Stab}_G(i) \simeq \mathrm{SO}_2(\mathbb{R})$, so that $\mathbb{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.
- $M = \Gamma \backslash \mathbb{H}$ for a *lattice* $\Gamma < \mathrm{SL}_2(\mathbb{R})$ (= discrete subgroup of finite co-volume).

Example: $SL_2(\mathbb{Z})\backslash\mathbb{H}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot z = z + 1$$ ("translation")

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \cdot z = -\frac{1}{z}$$ ("inversion")

together generate the lattice $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.



Classical Mechanics

Definition.

- *State of motion:* a possible *position* $z \in M$ and *velocity* $\vec{v} \in T_z^* M \ (\simeq \mathbb{R}^n)$
  Fact (Newton): given the current state $(z, \vec{v})$, there exists a unique state $(z', \vec{v}')$ the system will reach after t units of time.

- *Phase space* $T^* M \stackrel{\text{def}}{=} \{\text{all such pairs } (z, \vec{v})\}$.
  Also set $X = T^1 M = \left\{ (z, \vec{v}) \in T^* M \,\middle|\, \|\vec{v}\| = 1 \right\}$.

- *Observable*: a (smooth) function on phase space (i.e. $a \in C_c^\infty(T^* M)$)

- *Dynamics (Geodesic Flow):* $g_t : T^* M \to T^* M$ given by $g_t(z, \vec{v}) = (z', \vec{v}')$ from Newton.

Quantum Mechanics

Definition.

- *State of motion*: a function $\psi\colon M \to \mathbb{C}$ up to phase.
  Fact (Schrödinger): given the current state, we know the unique state the system will reach after t units of time.

- *Space of all states* is $L^2(M, d\,\mathrm{vol})$.

- Interpretation of ψ: prob. density for finding the particle given by

$$d\bar{\mu}_\psi(z) = \frac{1}{\|\psi\|_{L^2}^2} |\psi(z)|^2 \, d\,\mathrm{vol}_M(z).$$

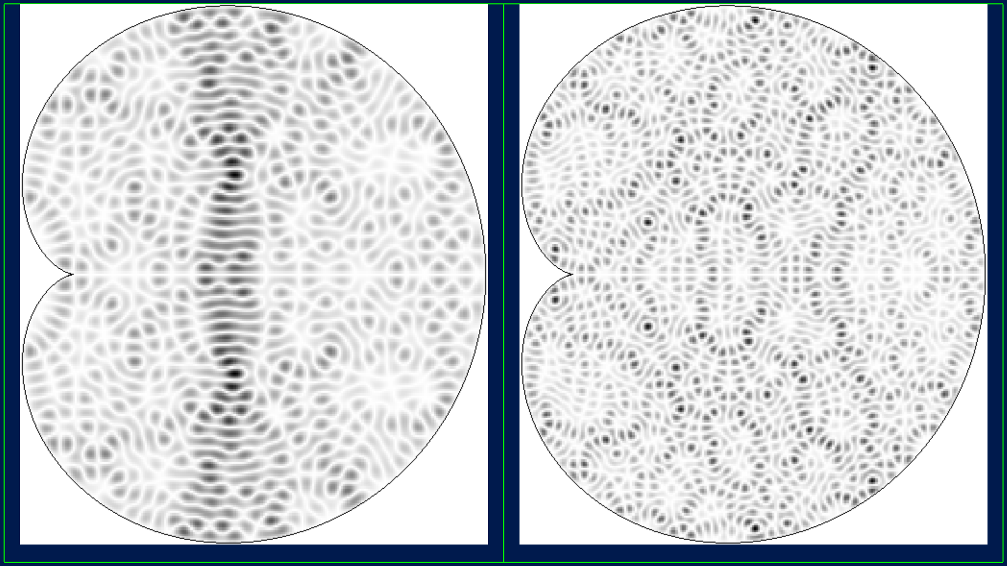
- *Observable*: operator $\mathrm{Op}\colon L^2(M) \to L^2(M)$.
  Takes definite value η if the state ψ satisfies $\mathrm{Op}\,\psi = \eta\psi$.

- Energy $\iff$ Laplace operator, with (orthonormal) system of eigenstates $\Delta\psi_n = -\lambda_n\psi_n$.
  Eigenstates are stationary, only linear combinations travel.

Examples of Eigenfunctions

On $M = (\mathbb{R}/\mathbb{Z})^2$, indexed by $k \in \mathbb{Z}^2$: $\psi_k(x) = e^{2\pi i k \cdot x}$ with $\lambda_k = -4\pi^2 \|k\|^2$. Note that $|\psi_k(x)|^2 = 1$.

On the Cardioid billiard (modes 567, 1277):



[A. Bäcker, arXiv:nlin.CD/0106018 & arXiv:nlin.CD/0204061]

Maass waveforms in the non-compact case.

In the example of $M = \Gamma\backslash\mathbb{H}$, assume $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ (true w.l.g. if M is non-compact).

- An eigenfunction $\psi\colon M \to \mathbb{R}$ is an Γ-periodic function on $\mathbb{H}$. Invariance under $z \mapsto z+1$ gives (Fourier expansion)

$$\psi(x+iy) = \sum_{n\in\mathbb{Z}} W_n(y) e^{2\pi i n x}.$$

Since $\Delta_{\mathbb{H}} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$,

$$\psi(x+iy) = \sum_{n\neq 0}^{\infty} a_n y^{1/2} K_{ir}\left(2\pi|n|y\right) e^{2\pi i n x},$$

where $\lambda = \frac{1}{4} + r^2$ and $K_{ir}(y)$ is the MacDonald-Bessel function. $a_0 = 0$ since ψ is square-integrable.

Hecke-Maass forms

Assume now that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (alternatively a *congruence subgroup*).

- We then have *Hecke Operators* $T_n \colon L^2(M) \to L^2(M)$

$$T_n\psi(z) = \frac{1}{\sqrt{n}} \sum_{\substack{a,d,b\,(n)\\ ad \equiv 1}} \psi\left(\frac{az+b}{d}\right),$$

  which commute with each other, with Δ, and with the reflection $T_{-1}\psi(z) = \psi(-\bar{z})$.

- For a joint eigenfunction ψ (a *Hecke-Maass form*) write $T_n\psi = \rho_\psi(n)\psi$. Then the Fourier coefficients are given by

$$a_n = a_1\rho_\psi(n).$$

- Corollary: the joint spectrum is simple.

Remark. A similar theory exists for certain compact quotients.

Remark. Based on numerical evidence it is expected that the spectrum of Δ on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (unlike other $\Gamma\backslash\mathbb{H}$) is already simple, in which case every Maass form would be a Hecke eigenform.

- Moreover, the associated L-function

$$L(s;\psi) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{a_n}{n^s} = a_1 \prod_{p \text{ prime}} \left(1 + \rho_\psi(p)p^{-s} - p^{1-2s}\right)^{-1}$$

  has good analytic properties, similar to those of the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}.$$

- We investigate analytic properties of generalizations of Hecke-Maass eigenforms.

The Semi-Classical Limit

- We know classical mechanics to be an accurate model in some situations, even though quantum mechanics is the underlying theory.

$\Rightarrow$ Expect the classical model to appear as a *limiting behaviour* of the quantum model.
("Correspondence Principle").

- Natural limit is that of *high energies* (in general, of $\hbar \to 0$).

Problem 1. What aspects of the classical system can be seen in the asymptotics of λ_n and ψ_n as $\lambda_n \to \infty$?

In particular, what can be said about the associated probability measures $\bar{\mu}_n$?

The Equidistribution Question

Fix an orthonormal basis of eigenfunctions $\{\psi_n\}_{n=1}^{\infty} \subset L^2(M)$ with $\lambda_0 \leq \lambda_1 \cdots$. For each observable $a \in C_c^{\infty}(M)$ (i.e. one such that $a(z, \vec{v}) = a(z)$ independently of $\vec{v}$) we have:

$$\bar{\mu}_n(a) \stackrel{\text{def}}{=} \int_M a(z) \left|\psi_n(z)\right|^2 d \operatorname{vol}_M(z).$$

Theorem. *(Schnirel'man-Zelditch-Colin de Verdière, "Quantum Ergodicity") ∃measures μ_n on $X = T^1 M$ lifting the $\bar{\mu}_n$ such that:*

1. *Every ("weak-*") limit μ_∞ of a subsequence of the μ_n is g_t-invariant.*
2. $\frac{1}{N+1} \sum_{n=0}^{N} \mu_n \xrightarrow[k \to \infty]{\text{wk-}*} \frac{d \operatorname{vol}_X}{\operatorname{vol}(X)}$.
3. *If the classical system is* ergodic *(almost every classical orbit is uniformly distributed) then $\mu_{n_k} \xrightarrow[k \to \infty]{\text{wk-}*} \frac{d \operatorname{vol}_X}{\operatorname{vol}(X)}$ along a subsequence of density one.*

Definition. The measures ν_n on X converge to the measure ν_∞ *in the weak-* topology* if for every observable $a \in C_c^\infty(X)$ we have $\lim_{n\to\infty} \nu_n(a) = \nu_\infty(a)$.

Definition. A g_t-invariant probability measure ν on X is *ergodic* if for every observable a and ν-almost every $x \in X$ we have:

$$\lim_{T\to\infty} \int_0^T a(g_t \cdot x)dt = \int_X a(x)d\nu(x).$$

In particular we call M (or X) *ergodic* if $d\,\mathrm{vol}_X$ is ergodic.

- Manifolds of negative curvature are g_t-ergodic (E. Hopf, A. Anosov).
- So are some plane billiards (e.g. the Cardioid).
- The flow on the torus is not ergodic (momentum is conserved!)

Example: Let $\gamma\colon [0,T] \to M$ be a closed geodesic. This can be lifted to a map $\tilde{\gamma} = (\gamma, \dot{\gamma})\colon [0,T] \to X$, and gives a g_t-invariant and ergodic *singular* measure $\nu_\gamma(a) = \frac{1}{T}\int_0^T a(\tilde{\gamma}(t))dt$.

Problem 2. What are the possible weak-* limits of $\{\mu_n\}_{n=1}^{\infty}$? ("Quantum Limits on X") When is the normalized volume measure the unique Quantum Limit?

- On completely integrable systems (e.g. the torus) there exists sequences of eigenfunctions which scar along every "regular" orbit (Toth-Zelditch).
- Naively, ergodicity would imply equidistribution since uncertainty would limit the localization.
- Numerical evidence indicates in some systems eigenfunctions become enhanced near periodic orbits. This phenomenon has been termed "scarring" by E. Heller.

Conjecture. *(Rudnick-Sarnak, "Quantum Unique Ergodicity") on compact manifolds of negative sectional curvature, the normalized volume measure is the unique quantum limit.*

The case of Hecke-Maass eigenforms is known as the question of *Arithmetic Quantum Unique Ergodicity.*

Other Problems

- Level spacing statistics. Weyl's law $N(\lambda) \overset{\text{def}}{=} \#\{n|\lambda_n \le \lambda\} = c(M)\lambda^{\dim M/2} + O(\lambda^{\dim M/2-1})$ gives the mean spacing.

$$\frac{\#\left\{\lambda_n \le \lambda \middle| a \le \frac{\lambda_n - \lambda_{n-1}}{\lambda^{\dim M/2}} \le b\right\}}{N(\lambda)} \xrightarrow[\lambda\to\infty]{} ?$$

- Value distribution.

- "Quantum Variance". When $\int_X a = 0$ Feingold-Perez conjecture

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \le \lambda} (\mu_n(a))^2 = \int_{-\infty}^{\infty} dt \, \langle a, a \circ g_t \rangle_X \, .$$

- Luo-Sarnak have shown that this fails in the arithmetic case.

- Recent numerical investigations by A. Barnett for an ergodic plane billiard.

Arithmetic Quantum Chaos

M a "congruence" surface, $\{\psi_n\}_{n=1}^{\infty} \subset L^2(M)$ the basis of Hecke-Maass eigenforms with $\lambda_0 = 0 < \lambda_1 \leq \cdots$.

Would like to show $\bar{\mu}_n(a) \to 0$ for $a \perp 1$. It suffices to check this for $a = \psi_k$.

Theorem. *("T. Watson's Formula")*

$$\frac{\left(\int_M \psi_k \psi_l \psi_m d\,\mathrm{vol}_M\right)^2}{\|\psi_k\|_{L^2}^2 \, \|\psi_l\|_{L^2}^2 \, \|\psi_m\|_{L^2}^2} = \star \frac{L(\frac{1}{2}; \psi_k \times \psi_l \times \psi_m)}{L(1; \wedge^2 \psi_k) L(1; \wedge^2 \psi_l) L(1; \wedge^2 \psi_l)}.$$

$\Rightarrow$"subconvexity" bounds toward the Grand Riemann Hypothesis for the triple product L-function would imply Arithmetic Quantum Unique Ergodicity in this case (+rate of equidistribution).

The Indirect Route

Would like to consider more general cases where no such formulas are expected.

- Think of the Hecke operators on $L^2(M)$ as arising from additional symmetries of M, not present for generic Γ.
- Hecke-Maass eigenforms ψ_n are the eigenstates which “respect” the symmetries.
- Analyze ψ_n using the symmetries.

$\Rightarrow$ Strategy for proving equidistribution (E. Lindenstrauss):

1. Lift: replace the measures $\bar{\mu}_n$ on M with related measures μ_n on a bundle $X \to M$, such that any limit is invariant under a flow $A \circlearrowright X$.

2. Additional smoothness: Show that any weak-* limit μ_∞ of the μ_n is not too singular.

3. Measure rigidity: Use results toward the classification of A-invariant measures on X to conclude that μ_∞ is the desired "uniform" measure.

When $M = \Gamma\backslash\mathbb{H} \simeq \Gamma\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, $T^1M \simeq \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ and the geodesic flow g_t is given by the action of the subgroup $A = \left\{\begin{pmatrix} e^{t/2} & \\ & e^{-t/2}\end{pmatrix}\right\}$ on the right.

- The Zelditch-Wolpert version of the Quantum Ergodicity theorem is compatible with the Hecke operators.
- J. Bourgain-Lindenstrauss: every $1 \neq a \in A$ acts on μ_∞ with *positive entropy*.
- Lindenstrauss: an A-invariant measure on $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ satisfying the positive entropy condition (and a recurrence condition) is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure.

Remark. If M is compact then it follows that μ_∞ is indeed the uniform measure. Otherwise we only know that $\mu_\infty = c \cdot d\,\mathrm{vol}_X$ for some constant $0 \leq c \leq 1$. To control this “escape of mass” a subconvexity bound on $L(\frac{1}{2}; \mathrm{Sym}^2\psi_n)$ would suffice.

Positive Entropy

We will control concentration on neighbourhood of geodesics.

Note that: $\left\{ x \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} \middle| |t| \leq \tau \right\}$ is a piece of length 2τ of the geodesic through x. Given $\varepsilon, \tau > 0$ we consider the tubular neighbourhood

$$B(\tau, \varepsilon) = \left\{ \begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} \middle| |u|, |v| \leq \epsilon, |t| \leq \tau \right\}.$$

We need to show: $\exists \kappa > 0$ such that for $\forall x \in X$,

$$\mu_\infty(xB(\varepsilon, \tau)) \leq C\varepsilon^\kappa \text{ as } \epsilon \to 0.$$

If X is non-compact the constant should be uniform on compact subsets $\Omega \subset X$. The uniform (“Lebesgue” or “Haar”) measure satisfies this with $\kappa = 2$.

The Hecke Correspondence

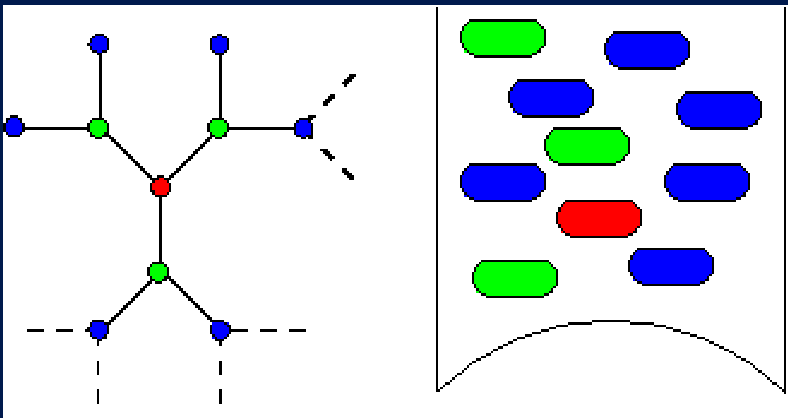
Alternative view of the Hecke Operators: given a prime p and $x \in X$, we have a subset $C_p(x) \subset X$ of size $p+1$ such that:

$$(T_p\psi)(x) = \frac{1}{\sqrt{p}} \sum_{x' \in C_p(x)} \psi(x').$$

- The relation $x \sim_p x' \iff x' \in C_p(x)$ is symmetric, giving a graph structure $G_p = (X, \sim_p)$.
- This is almost a $p+1$-regular *forest*: X is nearly a disjoint union of trees, and T_p is the "tree Laplacian".
  Problem: some components are not trees.
- This structure is equivariant w.r.t. the action of $\mathrm{SL}_2(\mathbb{R})$ on $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$.

We would like to show that $\mu_\infty\left(xB(\varepsilon,\tau)\right)$ is small.

Following Rudnick-Sarnak, Bourgain-Lindenstrauss: ($p = 2$ in the figure)



$\tilde{C}_p(x)$= sphere of rad. 2 ("blue vertices")

$C_p(x)$= sphere of rad. 1 ("green vertices")

For $g \in B(\varepsilon,\tau)$, $x' \in C_p(x)$ we have:

$$\rho_\psi(p)\psi(x'g) = \left(T_p\psi\right)\left(x'g\right) = \frac{1}{\sqrt{p}} \sum_{x''\in C_p(x')} \psi(x''g).$$

Summing over $x' \in C_p(x)$ and using $x \in C_p(x')$ one gets:

$$\psi(xg) = \frac{\sqrt{p}\rho_\psi(p)}{p+1} \sum_{x' \in C_p(x)} \psi(x'g) - \frac{1}{(p+1)} \sum_{x'' \in \tilde{C}_p(x')} \psi(x''g).$$

Now at least one of the two terms on the RHS must be as large as $\frac{1}{2}|\psi(xg)|$. Using Cauchy-Schwartz and assuming $|\rho_\psi(p)| \leq p^{\frac{1}{2}-\delta}$ gives:

$$\frac{C}{p^{1-2\delta}} |\psi(xg)|^2 \leq \sum_{x' \in C_p(x) \cup \tilde{C}_p(x)} |\psi(x'g)|^2.$$

Integrating over $g \in B(\varepsilon, \tau)$ gives:

$$\frac{C}{p^{1-2\delta}} \mu_\infty (xB(\varepsilon, \tau)) \leq \sum_{x' \in C_p(x) \cup \tilde{C}_p(x)} \mu_\infty (x'B(\varepsilon, \tau)).$$

Finally, we find a large set of primes $\mathcal{P}(x,\varepsilon)$ such that all the sets $\{x'B(\varepsilon,\tau) \mid p \in \mathcal{P}(x,\varepsilon), x \in C_p(x) \cup \tilde{C}_p(x)\}$ are *disjoint*. Then we have:

$$C\mu_\infty(xB(\varepsilon,\tau)) \sum_{p\in\mathcal{P}(x,\varepsilon)} \frac{1}{p^{1-2\delta}} \leq 1,$$

and hence:

$$\mu_\infty(xB(\varepsilon,\tau)) \leq C\left(\sum_{p\in\mathcal{P}(x,\varepsilon)} \frac{1}{p^{1-2\delta}}\right)^{-1}.$$

Difficult case: there is a closed geodesic nearby.

Bourgain-Lindenstrauss: replace $p^{-1+2\delta}$ by $\frac{3}{4}$; good choice of primes can make $\sum_{p\in\mathcal{P}(x,\varepsilon)} 1$ at least $\left(\frac{1}{\varepsilon}\right)^{2/9}$.

S-Venkatesh: good choice of correspondence and *all* primes $\leq \left(\frac{1}{\varepsilon}\right)^{\kappa}$. Get much smaller κ but method generalizes.

Higher-rank cases

G s.s. Lie Group	$\mathrm{SL}_n(\mathbb{R})$
$K < G$ max'l cpt. subgp	$\mathrm{SO}_n(\mathbb{R})$
$\Gamma < G$ congruence lattice	$\mathrm{SL}_n(\mathbb{Z})$
$M = \Gamma \backslash G / K$	loc. symm. space
$X = \Gamma \backslash G$	Weyl chamber bundle
ring of invariant differential ops.	includes Δ
$A < G$ max'l split torus	diagonal matrices

Maass forms are now $\psi \in L^2(M)$ which are joint eigenfunctions of the ring of G-invariant differential operators (isomorphic to $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$).

E.g. for $G = \mathrm{SL}_2(\mathbb{C})$, $K = \mathrm{SU}(2)$, G/K is hyperbolic 3-space.

Theorem 1. *(S-V) Let* $\{\psi_n\}_{n=1}^{\infty} \subset L^2(M)$ *be a non-degenerate sequence of eigenforms with* $\lambda_n \to \infty$*. Then there exists distribution* μ_n *on* X *lifting* $\bar{\mu}_n$ *such that every weak-* limit is* A*-invariant.*

Theorem 2. *(S-V) Assume* n *is prime, and* $\Gamma < \mathrm{SL}_n(\mathbb{R})$ *is a congruence lattice associated to a division algebra over* $\mathbb{Q}$*, split over* $\mathbb{R}$*. Then every regular element* $a \in A$ *acts on* μ_∞ *with positive entropy.*

Combined with measure rigidity results of Einsiedler-Katok-Lindenstrauss this shows that the unique non-degenerate quantum limit in that case is normalized Haar measure.