EFFECTIVE VERSIONS OF THE ČEBOTAREV DENSITY THEOREM

(PREPARED BY LIOR SILBERMAN)

Abstract. This set of notes is a rework of the paper [1] by Jeff Lagarias and Andrew Odlyzko, deriving explicit bounds on the error term in the density theorem. However, in this version I actually compute the constants.

GRH is assumed throughout!

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1. Introduction

Let K be an algebraic number field, L/K a normal Galois extension with group $G = \Gamma(L/K)$. Let d_L, d_K denote the absolute values of the discriminants of L, K. Let $n_L = [L : \mathbb{Q}], n_K = [K : \mathbb{Q}]$. $\mathfrak{p}$ will be a prime of K, $\mathcal{P}$ a prime of L. If $\mathfrak{p}$ is unramified in L, the Artin symbol $\left[\frac{L/K}{\mathfrak{p}}\right]$ denotes the conjugacy class of the Frobenius automorphisms corresponding to primes $\mathcal{P}|\mathfrak{p}$. We use $\mathrm{N} = \mathrm{N}_{L/K}$ to denote the usual norm.

Definition 1.1. Let C be a conjugacy class of G, x a positive number. We will estimate the function

$$\pi_C(x) = \pi_C(x, L/K) = \left|\left\{\mathfrak{p} \middle| \, \mathfrak{p} \text{ unramified in } L, \left[\frac{L/K}{\mathfrak{p}}\right] = C, \mathrm{N}\,\mathfrak{p} \le x\right\}\right|.$$

Recalling the familiar function

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x} \text{ as } x \to \infty$$

the density theorem asserts that:

$$\pi_C(x, L/K) \sim \frac{|C|}{|G|}\mathrm{Li}(x) \text{ as } x \to \infty.$$

More precisely, we will prove:

Theorem 1.2. *For $x \ge e^{10}$ we have:*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \le \frac{|C|}{|G|} 44 x^{1/2}\log d_L + \left(12.5\frac{\log x}{|G|} + 337\frac{|C|}{|G|}\right) x^{1/2} n_L + 2\log d_L.$$

Under the additional assumption:

Fact 1.3. *(Generalised Riemann Hypothesis) The nontrivial zeros of the Dedekind ζ-function ζ_L all lie on the line $\Im(\rho) = \frac{1}{2}$.*

It must be remarked here that the constants in the above estimate are fairly generous. It should be possible to reduce them significantly by making a more optimal choice of T (see proof of theorem 9.1) and by increasing $x_0 = e^{10}$. Also, the numbering of statements and equations here is independent of the numbering in the original paper. The sections do correspond, however.

2. Outline of the main argument

Along the lines of the usual proof of the prime number theorem (see, e.g. [2]), we define a weighted counting function $\psi_C(x) = \psi_C(x, L/K)$ by:

$$\psi_C(x, L/K) = \sum_{\substack{\mathrm{N}\,\mathfrak{p}^m \le x \\ \mathfrak{p} \text{ unramified} \\ \left[\frac{L/K}{\mathfrak{p}}\right]^m = C}} \log(\mathrm{N}\,\mathfrak{p}).$$

The standard program of attack is then:

1. $\psi_C(x)$ differs from a truncated inverse Mellin transform
$$I_C(x,T) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} F_C(s)\frac{x^s}{x}ds$$
by a remainder term $R_1(x,T)$.
2. $F_C(s)$ can be written as a linear combination of logarithmic derivatives of Hecke (Abelian) L-functions. Thus, any singularities of $F_C(S)$ are at most simple poles and occur at the zeros and pole of $\zeta_L(s)$.
3. "Shifting the line of integration of the left:" $I_C(x,T)$ differs from a contour integral $B_C(x,T) = \frac{1}{2\pi i}\oint_{B_T} F_C(s)\frac{x^s}{x}ds$ by a remainder term $R_2(x,T)$.
4. The contour integral is evaluated by the Cauchy's residuum theorem. The main term comes from the pole at $s = 1$ and is equal to $\frac{|C|}{|G|}x$, and the rest of the terms give a remainder $S(x,T)$ which is a sum over the zeros of $\zeta_L(s)$ within the contour B_T.
This leads to an explicit formula for $\psi_C(x)$ with several error terms, without using any conjectures.
5. $S(x,T)$ is estimated. We copy here only the GRH-based estimate.
6. The asymptotic formula $\psi_C(x) \sim \frac{|C|}{|G|}x$ with an explicit error term is derived by making an optimal choice of T.
7. The asymptotic formula $\pi_C(x) \sim \frac{|C|}{|G|}\,\mathrm{Li}(x)$ with an explicit error term is derived by partial summation.

3. Artin L-functions and Mellin Transforms

Let ϕ be a character of G.

Definition 3.1. Let $I_{\mathcal{P}} < G$ be the inertia group of $\mathcal{P}$, and let $e = |I|$. Let $\tau \in G$ be one of the Frobenius automorphisms corresponding to $\mathcal{P}$. We then write:

$$\phi_K(\mathfrak{p}^m) = \frac{1}{e}\sum_{\alpha\in I_{\mathcal{P}}} \phi(\tau^m\alpha). \tag{3.1}$$

It is easy to see that $\phi_K(\mathfrak{p}^m)$ is well-defined, i.e. independent of the choices of $\mathcal{P}$ and τ.

If $L(s,\phi,L/K)$ is the Artin L-series associated to ϕ, then for $\Re(s) > 1$:

$$-\frac{L'}{L}(s,\phi,L/K) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty} \phi_K(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-ms}. \tag{3.2}$$

Isolating the terms with $\mathfrak{p}^m \in C$ is done by picking a representative $g \in C$ and setting $f_C(\tau) = \sum_\chi \bar{\phi}(g)\phi(\tau)$ where the summation is over the irreducible characters of G. The orthogonality relations for characters imply that $f_C(\tau) = \frac{|G|}{|C|}1_C(\tau)$, and this means that if we define:

$$F_C(s) = -\frac{|C|}{|G|}\sum_{\phi} \bar{\phi}(g)\frac{L'}{L}(s,\phi,L/K) \tag{3.3}$$

then by (3.1) and (3.2) we have

$$F_C(s) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty} \theta(\mathfrak{p}^m)\log(\mathrm{N}\,\mathfrak{p})(\mathrm{N}\,\mathfrak{p})^{-ms}$$

where for $\mathfrak{p}$ unramified $\theta(\mathfrak{p}^m) = 1_C(\left[\frac{L/K}{\mathfrak{p}}\right]^m)$, and $|\theta(\mathfrak{p}^m)| \le 1$ still holds otherwise.

We now want to isolate the elements with small norm.

Lemma 3.2. *(discontinuous integral,* [2, pp. 105-106]*) Let $\Theta(y)$ be a (modified) Heaviside step-function: $\Theta(y) = 1$ for $y > 1$, $\Theta(y) = 0$ for $y < 1$ and $\Theta(y) = \frac{1}{2}$ for $y = 1$.*

Then for $y > 0, \sigma > 0$ and $T > 0$:
$\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{y^s}{s}ds - \Theta(y)\right| \le y^\sigma \min(1, T^{-1}|\log y|^{-1})$ *if* $y \ne 1$ *and* $\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{y^s}{s}ds - \Theta(y)\right| \le \sigma T^{-1}$ *if* $y = 1$.

Proof. The case $y = 1$ is obvious:
$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{y^s}{s}ds = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{ds}{s} = \frac{1}{2\pi i}\log\frac{\sigma+iT}{\sigma-iT} = \frac{1}{\pi}\tan^{-1}\frac{1}{\sqrt{1+(\sigma/T)^2}} \le \frac{\sigma}{T}$$
where we take the principal branch of the logarithm.

In the case $0 < y < 1$, $\lim_{\sigma\to\infty} \frac{y^s}{s} = 0$ uniformly in t ($s = \sigma + it$). This function is also analytic in the half-plane $\sigma > 0$. Considering the "contour" surrounding the "rectangle" $[\sigma, \infty) \times [-T, +T]$, we replace the integral by the sum

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\infty-iT} \frac{y^s}{s} ds - \frac{1}{2\pi i}\int_{\sigma+iT}^{\infty+iT} \frac{y^s}{s} ds$$

and since $\left|\int_{\sigma\pm iT}^{\infty\pm iT} \frac{y^s}{s} ds\right| \le \frac{1}{T}\int_\sigma^\infty y^s ds = \frac{y^\sigma}{T|\log y|}$ we in fact get the estimate $\frac{y^\sigma}{\pi T|\log y|}$.

We can also choose a circular arc centered at the origin with radius $R = \sqrt{\sigma^2 + T^2}$ as the path of integration (we take the short arc which does not enclose any poles). Then $|y^s| \le y^\sigma$ (note that $y < 1$) and $|s| = R$, so:

$$\frac{1}{2\pi}\left|\int_{\sigma-iT}^{\sigma+iT} \frac{y^s}{s} ds\right| \le \frac{y^\sigma}{2\pi R} 2\pi R = y^\sigma$$

(the last term is the length of the path of integration).

The case $y > 1$ is obtained by using the complementary arc of the circle and the "rectangle" $(-\infty, \sigma] \times [-T, +T]$, both of which include the pole at $s = 1$ with residue $1 (= \Theta(y))$. $\square$

Guided by the lemma, we define for $x \ge 2$:

$$I_C(x,T) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} F_C(s)\frac{x^s}{s} ds \tag{3.4}$$

where $\sigma_0 > 1$ will be chosen later.

The series (3.3) is absolutely convergent for $\Re(s) > 1$. Integrating term-by-term we have:

$$I_C(x,T) - \sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}\,\mathfrak{p}^m \le x}} \theta(\mathfrak{p}^m)\log \mathrm{N}\,\mathfrak{p} = \sum_{\mathfrak{p}, m}\left[\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right)^{-s}\frac{ds}{s} - \Theta\left(\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right)\right]\theta(\mathfrak{p}^m)\log \mathrm{N}\,\mathfrak{p} - \frac{1}{2}\sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}\,\mathfrak{p}^m = x}} \theta(\mathfrak{p}^m)\log \mathrm{N}\,\mathfrak{p}$$

where the last term is only present if there are $\mathfrak{p}$ and m such that $\mathrm{N}\,\mathfrak{p}^m = x$. Using the lemma, we then have:

$$\left|I_C(x,T) - \sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}\,\mathfrak{p}^m \le x}} \theta(\mathfrak{p}^m)\log \mathrm{N}\,\mathfrak{p}\right| \le \sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}\,\mathfrak{p}^m = x}}\left(\frac{1}{2}\log \mathrm{N}\,\mathfrak{p} + \sigma_0 T^{-1}\right) + R_0(x,T) \tag{3.5}$$

where

$$R_0(x,T) = \sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}\,\mathfrak{p}^m \ne x}}\left(\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right)^{\sigma_0}\min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right|^{-1}\right)\log \mathrm{N}\,\mathfrak{p}. \tag{3.6}$$

Now the sum in (3.5) is equal to $\psi_C(x)$ except for the terms coming from ramified primes, all of which divide d_L:

$$\left|\sum_{\substack{\mathfrak{p}, m \\ \mathrm{N}\,\mathfrak{p}^m \le x}} \theta(\mathfrak{p}^m)\log \mathrm{N}\,\mathfrak{p} - \psi_C(x)\right| \le \sum_{\substack{\mathfrak{p}, m \\ \mathfrak{p} \text{ ramified} \\ \mathrm{N}\,\mathfrak{p}^m \le x}} \log \mathrm{N}\,\mathfrak{p} \le \sum_{\mathfrak{p} \text{ ramified}} \log \mathrm{N}\,\mathfrak{p} \sum_{\mathrm{N}\,\mathfrak{p}^m \le x} 1 \le 2\log x \sum_{\mathfrak{p} \text{ ramified}} \log \mathrm{N}\,\mathfrak{p} \le 2\log x \log d_L$$

(and this estimate would also hold if C is the union of conjugacy classes of G). Also, $\mathrm{N}\,\mathfrak{p}^m = x$ means that $x = p^k$ for some rational prime p. Since there are at most n_K primes $\mathfrak{p}$ lying above p, $\sum_{\mathrm{N}\,\mathfrak{p}^m = x} \log \mathrm{N}\,\mathfrak{p} \le n_K \log x$ and we get:

$$\psi_C(x) = I_C(x,T) + R_1(x,T) \tag{3.7}$$

where

$$|R_1(x,T)| \le 2\log x \log d_L + \frac{1}{2}n_K \log x + n_K\sigma_0 T^{-1} + R_0(x,T). \tag{3.8}$$

We now estimate $R_0(x,T)$.

First of all, we make the choice $\sigma_0 = 1 + (\log x)^{-1}$, which is convenient since then $x^{\sigma_0} = ex$.

Secondly, we break the series R_0 into three parts. S_1 will consist of the terms for which $|\mathrm{N}\,\mathfrak{p}^m - x| \geq \frac{x}{4}$, S_2 of those for which $|\mathrm{N}\,\mathfrak{p}^m - x| \leq 1$, and S_3 of the remaining ones. If $|\mathrm{N}\,\mathfrak{p}^m - x| \geq \frac{x}{4}$ then $\left|\log \frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right| \geq \log\frac{5}{4}$ and

$$\min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}\,\mathfrak{p}^m}\right|^{-1}\right) \leq \frac{1}{\log 5/4}T^{-1}$$

for $T \geq \frac{1}{\log 5/4}$. Thus:

$$S_1 \leq \frac{exT^{-1}}{\log 5/4}\sum_{\mathfrak{p},m}(\mathrm{N}\,\mathfrak{p})^{-m\sigma_0}\log \mathrm{N}\,\mathfrak{p} = \frac{exT^{-1}}{\log 5/4}\left[-\frac{\zeta_K'}{\zeta_K}(\sigma_0)\right]. \tag{3.9}$$

For a final bound we use:

Lemma 3.3. *For $\sigma > 1$*

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) \leq -n_K\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma).$$

Proof. We have $-\frac{\zeta_K'}{\zeta_K}(\sigma) = \sum_{\mathfrak{p}}\frac{\log \mathrm{N}\,\mathfrak{p}}{(\mathrm{N}\,\mathfrak{p})^\sigma - 1}$. Also, for each prime $\mathfrak{p}$, $\mathrm{N}\,\mathfrak{p} = p^k$ for some rational prime p and some positive integer k. Thus

$$\frac{\log \mathrm{N}\,\mathfrak{p}}{(\mathrm{N}\,\mathfrak{p})^\sigma - 1} = \frac{k\log p}{p^{k\sigma}-1} = \frac{k}{\sum_{j=0}^{k-1}p^{j\sigma}}\cdot\frac{\log p}{p^\sigma - 1} \leq \frac{\log p}{p^\sigma - 1}.$$

Also, there are at most n_K distinct primes of K lying over a given rational prime p, so that:

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) \leq n_K\sum_p\frac{\log p}{p^\sigma - 1} = -n_K\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma).$$

□

Lemma 3.4. *For $1 < \sigma \leq 1 + \frac{1}{\log 2}$*

$$-\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma) \leq \frac{3}{\sigma - 1}.$$

Proof. Set $\varepsilon = \sigma - 1$, $\nu_0 = e^{1/\varepsilon}$ and $n_0 = \lfloor \nu_0 \rfloor$. Then:

$$\varepsilon\sum_{n\leq n_0}\frac{\log n}{n^\sigma} \leq \sum_{n\leq n_0}\frac{1}{n^\sigma}.$$

Also, for $\nu \geq e^{1/\sigma}$, $\frac{\log\nu}{\nu^\sigma}$ is a monotone decreasing (compute the derivative) and $\varepsilon^{1/\sigma} \leq n_0$ for our range. Integrating by parts gives:

$$\varepsilon\sum_{n>n_0}\frac{\log n}{n^\sigma} \leq \varepsilon\int_{n_0}^\infty\frac{\log\nu}{\nu^\sigma}d\nu = \varepsilon\left[-\frac{\log\nu}{\varepsilon\nu^\varepsilon}\right]_{n_0}^\infty - \varepsilon\left[\frac{1}{\varepsilon^2\nu^\varepsilon}\right]_{n_0}^\infty = \frac{\varepsilon\log n_0 + 1}{\varepsilon n_0^\varepsilon} \leq \frac{2}{\varepsilon n_0^\varepsilon}$$

and on the other hand,

$$\sum_{n>n_0}\frac{1}{n^\sigma} \geq \int_{n_0+1}^\infty\frac{d\nu}{\nu^\sigma} = \frac{1}{\varepsilon(n_0+1)^\varepsilon}.$$

We combine the two, and get:

$$\varepsilon\sum_{n=1}^\infty\frac{\log n}{n^\sigma} \leq \max\left\{1, 2\left(\frac{n_0+1}{n_0}\right)^\varepsilon\right\}\sum_{n=1}^\infty\frac{1}{n^\sigma}.$$

To conclude to proof, it suffices to estimate the constant inside. Since $\nu_0 - 1 < n_0 < n_0 + 1 < \nu_0 + 1$,

$$2\left(\frac{n_0+1}{n_0}\right)^\varepsilon \leq 2\left(\coth\frac{1}{2\varepsilon}\right)^\varepsilon = 2\left[\left(\coth\frac{1}{2\varepsilon}\right)^{2\varepsilon}\right]^{1/2}.$$

However, $(\coth x)^x \leq 1.5$ for all $x > 0$ and we are done. □

Corollary 3.5. *For $1 < \sigma \leq 1 + \frac{1}{\log 2}$ we have $-\frac{\zeta_K'}{\zeta_K}(\sigma) \leq \frac{3n_K}{\sigma - 1}$.*

Applying this to (3.9) leads to:

$$S_1 \leq \frac{exT^{-1}}{\log 5/4}\left[-\frac{\zeta_K'}{\zeta_K}(\sigma_0)\right] \leq \frac{exT^{-1}}{\log 5/4} n_K \frac{3}{\sigma_0 - 1} = \frac{3e}{\log 5/4} T^{-1} x \log x. \tag{3.10}$$

S_2 consists of terms $\mathfrak{p}^m$ for which $0 < |\operatorname{N}\mathfrak{p}^m - x| \leq 1$. There at most $2n_K$ such terms leading to the simple bound

$$S_2 \leq 2n_K \left(\frac{x}{x-1}\right)^{\sigma_0} \log(x+1)$$

which for $x \geq e^6$ can be rewritten as

$$S_2 \leq 2.01 n_K \log x. \tag{3.11}$$

The terms in S_3 satisfy $1 < |\operatorname{N}\mathfrak{p}^m - x| < \frac{x}{4}$ and thus:

$$S_3 \leq T^{-1}\left(\frac{5}{4}\right)^{\sigma_0} (\log\frac{5}{4} + \log x) \sum_{1<|n-x|<\frac{x}{4}} \left|\log\frac{x}{n}\right|^{-1} \sum_{\operatorname{N}\mathfrak{p}^m = n} 1$$

since $n \geq \frac{x}{2}$ the estimate $\left|\log\frac{x}{n}\right|^{-1} \leq \frac{2n}{|x-n|}$ holds and

$$S_3 \leq n_K T^{-1}\left(\frac{5}{4}\right)^{\sigma_0} (\log\frac{5}{4} + \log x)(5x) \sum_{k<\frac{x}{4}} \frac{1}{k}.$$

Using the estimate $\sum_{k<y} \frac{1}{k} \leq 1 + \log(y/2)$ we write this as:

$$S_3 \leq 5 n_K x T^{-1} \log^2 x. \tag{3.12}$$

Combining (3.10)-(3.12) gives:

$$R_0(x,T) \leq 5n_K T^{-1} x \log^2 x + 36.55 n_K T^{-1} x \log x + 2.01 n_K \log x \tag{3.13}$$

And using this in (3.8) gives:

$$R_1(x,T) \leq 2\log x \log d_L + 2.51 n_K \log x + 5n_K x T^{-1} \log^2 x + 36.55 n_K T^{-1} x \log x, \tag{3.14}$$

valid for $x \geq e^6, T \geq 1$.

4. Reduction to the case of Hecke L-functions

The definition of $F_C(s)$ in equation (3.3) is in terms of Artin L-functions corresponding to the (usually nonlinear) characters of $G(L/K)$. This section uses a reduction due to Deuring (see [3]) to the case of Hecke (Abelian) L-functions which allows better estimates on the location and density of the singularities of $F_C(s)$. The following formulation is due to Serre.

That definition used a representative $g \in C$. Let $H = \langle g \rangle$ be the cyclic group generated by g, M the fixed field of H. χ will denote the irreducible characters of H (which are 1-dimensional).

Lemma 4.1. *We have*

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L}(s, \chi, L/M).$$

Proof. Let $f : H \mapsto \mathbb{C}$ be the class function $f = \sum_\rho \bar{\chi}(g)\chi$. Since H is Abelian, the orthogonality relations now give $f(h) = |H| \cdot \delta_{h,g}$ (i.e. the indicator at g). The class function on G induced by f is obviously $\operatorname{Ind}_H^G f(y) = |C_G(g)| \cdot 1_C(y)$ and since $|C| = [G : C_G(g)]$, we have $\operatorname{Ind}_H^G f = f_C$. Therefore:

$$\sum_\chi \bar{\chi}(g) \operatorname{Ind}_H^G \chi = \sum_\phi \bar{\phi}(g)\phi$$

and for $\Re(s) > 1$ this means that:

$$F_C(s) = -\frac{|C|}{|G|} \sum_\chi \bar{\chi}(g) \frac{L'}{L}(s, \operatorname{Ind}_H^G \chi, L/K)$$

But $L(s, \operatorname{Ind}_H^G \chi, L/K) = L(s, \chi, L/M)$ proves the desired relation for $\Re(s) > 1$. By analytic continuation we are done. ∎

5. Density of Zeros of Hecke L-functions

The results of section 3 for the extension L/M now show that:

$$\psi_C(x) = I_C(x,T) + R_1(x,T)$$

where $|R_1(x,T)|$ satisfies (3.14) and

$$I_C(x,T) = -\frac{1}{2\pi i}\frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{x^s}{s}\frac{L'}{L}(s,\chi,L/M)ds \tag{5.1}$$

where $\sigma_0 = 1 + (\log x)^{-1}$ and χ runs through the (1-dimensional) characters of $H = \langle g\rangle$. The next goal is to evaluate the integrals (5.1), which turns out to be more convenient than integrating $F_C(s)$ directly. This requires bounds on the singularities of $\frac{L'}{L}$.

Fixing the fields L and M from now on, $L(s,\chi)$ will denote $L(s,\chi,L/M)$. $F(\chi)$ will denote the conductor of χ, and we set:

$$A(\chi) = d_M\,\mathrm{N}_{\mathbb{Q}}^{M}(F(\chi))$$

and define $\delta(\chi)$ to be 1 if χ is the principal character, 0 otherwise.

Recalling the functional equation for Hecke L-functions, we know that there exist non-negative integers $a = a(\chi)$ and $b = b(\chi)$ such that $a + b = n_E$ and such that if we define (the Euler factor at the infinite primes)

$$\gamma_\chi(s) = \left[\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right]^b\left[\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right]^a$$

and (the completed L-function)

$$\xi(s,\chi) = [s(s-1)]^{\delta(\chi)}A(\chi)^{s/2}\gamma_\chi(s)L(s,\chi)$$

then $\xi(s,\chi)$ will satisfy the functional equation

$$\xi(1-s,\bar{\chi}) = W(\chi)\xi(s,\chi) \tag{5.2}$$

with some constant $W(\chi)$ of absolute value 1.

Furthermore, $\xi(s,\chi)$ is an entire function of order 1, and can therefore be written as:

$$\xi(s,\chi) = e^{B_1(\chi)+B(\chi)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$

where ρ runs through all the zeroes of $\xi(s,\chi)$, which are precisely the zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ for which $0 < \beta < 1$ (the so-called *nontrivial zeros)* [recall that $L(s,\chi)$ and therefore $\xi(s,\chi)$ has no zeroes for $s \geq 1$]. $B(\chi)$ and $B_1(\chi)$ are constants.

Computing the logarithmic derivatives and accounting for the infinite Euler factor gives:

$$\frac{L'}{L}(s,\chi) = B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) - \frac{1}{2}\log A(\chi) - \delta(\chi)\left[\frac{1}{s}+\frac{1}{s-1}\right] - \frac{\gamma_\chi'}{\gamma_\chi}(s) \tag{5.3}$$

as an equality of functions meromorphic in the complex plane.

$B(\chi)$ presents a problem – its dependence on χ has yet to be determined.

We use the fact that χ is an Abelian character, and that therefore $(s-1)^{\delta(\chi)}L(s,\chi)$ is entire and satisfies a functional equation. The following lemma is adapted from Lemma 2 of [4]:

Lemma 5.1. *Suppose that $(s-1)^r L(s,\chi)$ is entire for some non-negative integer r, minimal with this property. Then:*

$$\log A(\chi) = -2G(a,b,s) - \frac{L'}{L}(s,\chi) - \frac{L'}{L}(s,\bar{\chi}) + \sum_{\rho}\left[\frac{1}{s-\rho}+\frac{1}{s-\bar{\rho}}\right] - \frac{2r}{s-1} - \frac{2r}{s}$$

Identically in s, where ρ runs through the nontrivial zeros of $L(s,\chi)$.
(the notation is defined later)

Proof. The completed L-function is defined by:

$$\xi(s,\chi) = A^{s/2}\Gamma\left(\frac{s}{2}\right)^a\Gamma\left(\frac{s+1}{2}\right)^b L(s,\chi)$$

for some constants $A(\chi), a(\chi), b(\chi)$ (the last two are non-negative integers) which are the same if χ is replaced by $\bar{\chi}$.

Since $(s-1)^r L(s,\chi)$ is entire, so is $(s-1)^r L(s,\bar{\chi})$ and thus if we set:

$$f(s) = [s(s-1)]^{2r}\xi(s,\chi)\xi(s,\bar{\chi})$$

then $f(s)$ is entire as well, and its functional equation reads:

$$f(1-s) = W(\chi)W(\bar{\chi})f(s). \tag{5.4}$$

Furthermore, $\bar{f}(s) = f(\bar{s})$, so that if z is a zero of $f(s)$, then so are $\bar{z}, 1-\bar{z}$ and $1-z$. Also, since the zeros of $f(s)$ are precisely the nontrivial zeros of the two L-functions, they all lie on the critical strip.

Since the L-functions are of order 1, so if f, and we can write:

$$f(s) = e^{A+Bs}\prod_{\rho}\left[\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{\bar{\rho}}\right)e^{s/\rho+s/\bar{\rho}}\right]$$

($f(0)\neq 0$ by the minimality of r) and hence:

$$\frac{f'}{f}(s) = B + \sum_{\rho}\left[\frac{1}{s-\rho}+\frac{1}{s-\bar{\rho}}+\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right].$$

By the functional equation (5.4), $\frac{f'}{f}(s) = -\frac{f'}{f}(1-s)$, which means:

$$B + \sum_{\rho}\left[\frac{1}{s-\rho}+\frac{1}{s-\bar{\rho}}+\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right] = -B - \sum_{\rho}\left[\frac{1}{1-s-\rho}+\frac{1}{1-s-\bar{\rho}}+\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right]$$

or:

$$B = -\sum_{\rho}\left[\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right] + \frac{1}{2}\sum_{\rho}\left[\frac{1}{\rho-s}+\frac{1}{\bar{\rho}-s}-\frac{1}{1-\rho-s}-\frac{1}{1-\bar{\rho}-s}\right].$$

However, since $1-\bar{\rho}$ is a nontrivial zero of $L(s,\chi)$ whenever ρ is (including multiplicities), the second term cancels out and we have:

$$B = -\sum_{\rho}\left[\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right]$$

and

$$\frac{f'}{f}(s) = \sum_{\rho}\left[\frac{1}{s-\rho}+\frac{1}{s-\bar{\rho}}\right].$$

The lemma follows immediately, where we see that $G(a,b,s) = a\frac{d}{ds}\log\Gamma(\frac{s}{2}) + b\frac{d}{ds}\log\Gamma(\frac{s+1}{2})$. □

In particular, since the B in the lemma satisfies $B(\chi)+B(\bar{\chi}) = B(\chi)+\bar{B}(\chi) = 2\Re(B(\chi))$ we have:

$$\Re B(\chi) = -\sum_{\rho}\Re\frac{1}{\rho}$$

and (note the slight difference in the definition of $A(\chi)$ - we switch back to the one used *before* the lemma).

$$\frac{L'}{L}(s,\chi)+\frac{L'}{L}(s,\bar{\chi}) = \sum_{\rho}\left[\frac{1}{s-\rho}+\frac{1}{s-\bar{\rho}}\right] - \log A(\chi) - 2\delta(\chi)(\frac{1}{s}+\frac{1}{s-1}) - 2\frac{\gamma_\chi'}{\gamma_\chi}(s). \tag{5.5}$$

This lemma gives estimates of both B and the density of zeroes of $L(s,\chi)$. An analogue could be proven for Artin L-functions, but there would then be terms for the possible poles (except at 0) and that would weaken the estimates. This was avoided by reducing to the case of abelian L-functions.

We now write $\sigma = \Re(s)$ and estimate the various parts of the identity.

Lemma 5.2. *For $\sigma \geq \frac{1}{\log 2}$, $\varepsilon = \sigma - 1$, we have:*

$$-\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma) \leq \frac{\varepsilon\log 2+1}{\varepsilon\sigma 2^{\varepsilon}}.$$

Proof. In the present case, $\frac{\log \nu}{\nu^\sigma}$monotone decreasing function for $\nu \geq 2$ (and there is no $n = 1$ term, of course). Therefore:

$$\sum_{n=1}^{\infty} \frac{\log n}{n^\sigma} \leq \int_2^\infty \frac{\log \nu}{\nu^\sigma} d\nu = \left[-\frac{\log \nu}{\varepsilon \nu^\varepsilon}\right]_2^\infty - \left[\frac{1}{\varepsilon^2 \nu^\varepsilon}\right]_2^\infty = \frac{\varepsilon \log 2 + 1}{\varepsilon^2 2^\varepsilon}.$$

Also, $\frac{1}{\nu^\sigma}$ is obviously monotone decreasing so

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} \geq 1 + \int_1^\infty \frac{1}{\nu^\sigma} d\nu = \left[-\frac{1}{\varepsilon \nu^\varepsilon}\right]_1^\infty = 1 + \frac{1}{\varepsilon} = \frac{\sigma}{\varepsilon}.$$

□

Dividing the two gives the desired result.

Corollary 5.3. *If $\sigma \geq \frac{1}{\log 2}$ then*

$$\left|\frac{L'}{L}(s, \chi)\right| < n_M \frac{\varepsilon \log 2 + 1}{\varepsilon(\varepsilon + 1)2^\varepsilon}$$

Proof. A comparison of the Dirichlet series shows that

$$\left|\frac{L'}{L}(s, \chi)\right| \leq -\frac{\zeta_M'}{\zeta_M}(\sigma)$$

and we saw beforehand that $-\frac{\zeta_M'}{\zeta_M}(\sigma) \leq -n_M \frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(\sigma)$.

We now compute some estimates for the function $F(s) \stackrel{\text{def}}{=} \frac{\Gamma'}{\Gamma}(s)$. □

Lemma 5.4. *If $\sigma \geq 1$ then*

$$|F(s)| \leq \gamma + 3 \log |s|$$

Proof. The Weierstrass product of the Gamma function is:

$$\frac{1}{s\Gamma(s)} = e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n}$$

from which we deduce:

$$F(s) = -\frac{1}{s} - \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{s+n}\right)$$

and also:

$$F'(s) = \sum_{n=0}^{\infty} \frac{1}{(n+s)^2}.$$

Now, $|s| \geq \sigma \geq 1$ and thus $m = \lfloor |s| \rfloor \geq 1$. Also, $|s| \leq |n + s| \leq |n| + |s|$. Thus:

$$|F'(s)| \leq \frac{m+1}{|s|^2} + \sum_{n=m+1}^{\infty} \frac{1}{(n+\sigma)(n+\sigma-1)} = \frac{m+1}{|s|^2} + \frac{1}{m+\sigma} \leq \frac{2}{|s|} + \frac{1}{|s|^2} \leq \frac{3}{|s|}.$$

Finally,

$$|F(s)| \leq |F(1)| + \int_1^{|s|} |F'(z)| \cdot d|z| \leq |-\gamma| + \int_1^{|s|} \frac{3dx}{x} = \gamma + 3 \log |s|.$$

□

We now extend the estimates to cover smaller values of $\Re(s)$.

Lemma 5.5. *If $|z + k| \geq 1/8$ for all non-negative integers k, then*

$$|F(z)| \leq 3\gamma + 3 \log 2 + \frac{1}{4} + 5 \log(|z| + 1).$$

Proof. If $\Re(z) \geq 1$ this is weaker than the previous lemma. Otherwise, we use the functional equation:

$$\digamma(u+1) = \digamma(u) + \frac{1}{u}$$

(equivalent to $\Gamma(u+1) = u\Gamma(u)$), to show that:

$$\digamma(z) = \digamma(z+m) - \sum_{k=0}^{m-1} \frac{1}{z+k}$$

for any positive integer m. For $m = \lceil |z| \rceil + 1$, $\Re(z+m) > 1$ and we have

$$|\digamma(z+m)| \leq \gamma + 3\log(|z+m|).$$

From $|z+m| \leq 2|z| + 2 = 2(|z|+1)$ we conclude

$$|\digamma(z+m)| \leq \gamma + 3\log 2 + 3\log(|z|+1). \tag{5.6}$$

As to the sum, we separate it into the parts where, and get:

$$\left|\sum_{k=0}^{m-1} \frac{1}{z+k}\right| \leq \sum_{k=0}^{m-1} \left|\frac{1}{z+k}\right| \leq \frac{2}{8} + \sum_{\Re(z+k)\geq 1}^{k \leq m-1} \frac{1}{\Re(z+k)} - \sum_{k \geq 0}^{\Re(z+k)\leq -1} \frac{1}{\Re(z+k)}.$$

We now note that (for the first sum) $\Re(z+k) \geq \lfloor z+k \rfloor$ and that the last summand is at most $\lceil |z| \rceil$. Similarly for the other sum, so:

$$\left|\sum_{k=0}^{m-1} \frac{1}{z+k}\right| \leq \frac{1}{4} + \sum_{k=1}^{\lceil |z| \rceil} \frac{1}{k} + \sum_{k=1}^{\lceil |z| \rceil} \frac{1}{k} \leq \frac{1}{4} + 2\gamma + 2\log(|z|+1)$$

which when combined with equation (5.6) concludes the proof. □

Lemma 5.6. *If $\sigma \geq 2$ then*

$$\left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right| \leq \frac{n_M}{2}\left[\log\pi + \gamma + 3\log\left(|s|+1\right)\right].$$

Proof. From the definition of γ_χ, we have:

$$\log\gamma_\chi(s) = -\frac{1}{2}\log\pi - \frac{a+b}{2}s\log\pi + a\log\Gamma(\frac{s}{2}) + b\log\Gamma(\frac{s+1}{2})$$

and therefore (using $a + b = n_M$)

$$2\left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right| \leq n_M\log\pi + n_M\left|\digamma(\frac{s+1}{2})\right|.$$

Using lemma (5.4) the desired result follows. □

Now let $n_\chi(t)$ denote the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $0 < \beta < 1$, $|\gamma - t| \leq 1$.

Lemma 5.7. *For all t, we have:*

$$n_\chi(t) \leq 5\log A(\chi) + 15n_M\left[3 + \log\left(|t|+1\right)\right]$$

Proof. We evaluate (5.5) at $s = 2 + it$. Lemmas (5.3) and (5.6) imply that:

$$\sum_\rho \Re\left[\frac{1}{s-\rho} + \frac{1}{s-\bar{\rho}}\right] \leq 2n_M\frac{\log 2 + 1}{4} + \log A(\chi) + 3 + n_M\left[\log\pi + \gamma + 3\log\left(|s|+1\right)\right]$$

and since $\frac{t+1}{1+\sqrt{4+t^2}}$ is monotone increasing for $t \geq 0$, and thus at least $\frac{1}{3}$, we have $|s| + 1 \leq 3(|t|+1)$ and:

$$\sum_\rho \Re\left[\frac{1}{s-\rho} + \frac{1}{s-\bar{\rho}}\right] \leq \log A(\chi) + 3 + n_M\left[\log\pi + \gamma + 3\log 3 + \frac{1+\log 2}{2} + 3\log\left(|t|+1\right)\right]$$

on the other hand $2 = \Re(s) > \Re(\rho)$ implies $\Re(s-\rho)^{-1}, \Re(s-\bar{\rho})^{-1} > 0$ and then:

$$\sum_\rho \Re\left[\frac{1}{s-\rho} + \frac{1}{s-\bar{\rho}}\right] \geq \sum_{|\gamma-t|\leq 1} \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2} \geq \sum_{|\gamma-t|\leq 1} \frac{1}{5} = \frac{1}{5}n_\chi(t)$$

since $1 < 2 - \beta < 2$, which proves the lemma. □

In a similar fashion, we obtain:

Lemma 5.8. *For any $0 < \varepsilon \leq 1$ we have:*

$$\left| B(\chi) + \sum_{|\rho|<\varepsilon} \frac{1}{\rho} \right| \leq 258.36 n_M + 22 \log A(\chi) + 5\varepsilon^{-1}(\log A(\chi) + 9n_M)$$

Proof. We evaluate (5.3) at $s = 2$. Lemmas (5.3) and (5.6) imply that:

$$\left| B(\chi) + \sum_{\rho} \left(\frac{1}{2-\rho} + \frac{1}{\rho} \right) \right| \leq n_M \frac{\log 2 + 1}{4} + \frac{1}{2} \log A(\chi) + \frac{3}{2} + \frac{n_M}{2} \left[\log \pi + \gamma + 3 \log 3 \right]$$

Now $\left| \frac{1}{2-\rho} + \frac{1}{\rho} \right| = \frac{2}{|\rho(2-\rho)|} \leq \frac{2}{|\rho|^2}$ and therefore the previous lemma implies

$$\sum_{|\rho|\geq 1} \left| \frac{1}{2-\rho} + \frac{1}{\rho} \right| \leq 2 \sum_{j=1}^{\infty} \frac{n_\chi(j)}{j^2} \leq 10 \left[\log A(\chi) + 9n_M \right] \sum_{j=1}^{\infty} \frac{1}{j^2} + 30 n_M \sum_{j=1}^{\infty} \frac{\log(j+1)}{j^2}$$

or

$$\sum_{|\rho|\geq 1} \left| \frac{1}{2-\rho} + \frac{1}{\rho} \right| \leq \frac{5\pi^2}{3} \log A(\chi) + 15 \left[\pi^2 + 6 \log 2 \right] n_M$$

Also, $|2 - \rho| \geq 1$ so

$$\sum_{|\rho|<1} \left| \frac{1}{2-\rho} \right| \leq n_\chi(0) \leq 5 \log A(\chi) + 45 n_M$$

and also

$$\sum_{\varepsilon \leq |\rho| < 1} \frac{1}{|\rho|} \leq \varepsilon^{-1} n_\chi(0)$$

We now combine these estimates to get the desired result. □

Corollary 5.9. *Assuming the GRH as well,*

$$|B(\chi)| \leq 348.36 n_M + 32 \log A(\chi)$$

Proof. Take $\varepsilon = \frac{1}{2}$ in the previous lemma – under the assumption there are no roots with $|\rho| < \varepsilon$. □

Lemma 5.10. *If $\sigma > -1/2$ and $|s| \geq 1/8$ then*

$$\left| \frac{\gamma_\chi'}{\gamma_\chi}(s) \right| \leq 17.5 n_M + 1.5 n_M \log(|s| + 4).$$

Proof. From the definition of γ_χ, we have:

$$\log \gamma_\chi(s) = -\frac{1}{2} \log \pi - \frac{a+b}{2} s \log \pi + a \log \Gamma(\frac{s}{2}) + b \log \Gamma(\frac{s+1}{2})$$

and therefore (using $a + b = n_M$)

$$2 \left| \frac{\gamma_\chi'}{\gamma_\chi}(s) \right| \leq n_M \log \pi + a|F(\frac{s}{2})| + b|F(\frac{s+1}{2})|. \tag{5.7}$$

We now estimate $|F(\frac{s}{2})|$ and $|F(\frac{s+1}{2})|$ using lemma (5.4) (and $\Re(\frac{s+1}{2}) \geq 1/4$):

$$|F(\frac{s}{2})| \leq |\frac{2}{s}| + |\frac{2}{s+2}| + |F(\frac{s}{2} + 2)| \leq 16 + \frac{4}{3} + \gamma + 3 \log |\frac{s}{2} + 2|,$$

$$|F(\frac{s+1}{2})| \leq |\frac{2}{s+1}| + \gamma + \log |\frac{s+1}{2} + 1| \leq 4 + \gamma + 3 \log |\frac{s+3}{2}|.$$

Combining the estimates gives:

$$2 \left| \frac{\gamma_\chi'}{\gamma_\chi}(s) \right| \leq n_M \log \pi + 18 n_M - 3 n_M \log 2 + 3 n_M \log(|s| + 4).$$

□

Lemma 5.11. *If $-1/2 \le \sigma \le 3$, $|s| \ge 1/8$ and $t \ge 2$ then:*

$$\left|\frac{L'}{L}(s,\chi)+\frac{\delta(\chi)}{s-1}-\sum_{|\gamma-t|\le 1}\frac{1}{s-\rho}\right| \le 2585n_M+1510.5n_M\log(|t|+7)+331.5\log A(\chi)$$

Proof. We evaluate (5.3) at $s=\sigma+it$ and $3+it$ and subtract the results [eliminating $B(\chi)$] to obtain:

$$\frac{L'}{L}(s,\chi)-\frac{L'}{L}(3+it,\chi)=\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{3+it-\rho}\right)-\frac{\gamma_\chi'}{\gamma_\chi}(s)+\frac{\gamma_\chi'}{\gamma_\chi}(3+it)-\delta(\chi)\left(\frac{1}{s}+\frac{1}{s-1}-\frac{1}{2+it}-\frac{1}{3+it}\right).$$

We now estimate $L(3+it,\chi)$ and the gamma factors using lemmas (5.3), (5.10) and (5.6) respectively to get:

$$\left|\frac{L'}{L}(s,\chi)+\frac{\delta(\chi)}{s-1}-\sum_{|\gamma-t|\le 1}\frac{1}{s-\rho}\right|\le 18.5n_M+3n_M\log(|t|+7)+\sum_{|\gamma-t|>1}\left|\frac{1}{s-\rho}-\frac{1}{3+it-\rho}\right|+\sum_{|\gamma-t|\le 1}\left|\frac{1}{3+it-\rho}\right|$$

since $|3+it-\rho|\ge 2$ we can bound the last sum by $\frac{1}{2}n_\chi(t)$ (see lemma (5.7)). For the other one,

$$\sum_{|\gamma-t|>1}\left|\frac{1}{s-\rho}-\frac{1}{3+it-\rho}\right|=\sum_{|\gamma-t|>1}\frac{3-\sigma}{|s-\rho||3+it-\rho|}\le 20\sum_{j=1}^{\infty}\frac{n_\chi(t+j)+n_\chi(t-j)}{j^2}\le\frac{100\pi^2}{3}\left[\log A(\chi)+9n_M\right]+600n_M\sum_{j=1}^{\infty}\frac{\log(|t|+j+1)}{j^2}.$$

We can bound the last term by:

$$\sum_{j=1}^{\infty}\frac{\log(|t|+j+1)}{j^2}\le\log(|t|+2)+\int_2^\infty\frac{\log(\lambda+|t|+1)}{\lambda^2}d\lambda=\log(|t|+2)+\frac{3}{2}\log(|t|+3)-\log 2$$

since for $u=|t|+1\ge 3$ and $x\ge 2$, $\frac{\log(x+u)}{x^2}$is monotone decreasing. □

6. The contour integral

The next step is to evaluate $I_C(x,T)$ by evaluating:

$$I_\chi(x,T)=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{x^s}{s}\frac{L'}{L}(s,\chi)ds$$

for each character χ of H. In addition to $T\ge 1$, we now assume T does not coincide with the ordinate of a zero of any of the $L(s,\chi)$. Also, let U be a new parameter, where $U=j+\frac{1}{2}$ for some non-negative integer j (later, $U\to\infty$). Let $B_{T,U}$ be the positively oriented rectangle with vertices at $\sigma_0-iT,\sigma_0+iT,-U+iT,-U-iT$. The contour integral

$$I_\chi(x,T,U)=\frac{1}{2\pi i}\oint_{B_{T,U}}\frac{x^s}{s}\frac{L'}{L}(s,\chi)ds \tag{6.1}$$

can be readily evaluated in terms of the singularities of the integrand. We here show that the difference $R_\chi(x,T,U)=I_\chi(x,T,U)-I_\chi(x,T)$ is small.

This remainder may be divided into the vertical integral

$$V_\chi(x,T,U)=\frac{1}{2\pi}\int_{+T}^{-T}\frac{x^{-U+it}}{-U+it}\frac{L'}{L}(-U+it,\chi)dt$$

and two horizontal integrals:

$$H_\chi(x,T,U)=\frac{1}{2\pi i}\int_{-U}^{-1/4}\left\{\frac{x^{\sigma-iT}}{\sigma-iT}\frac{L'}{L}(\sigma-iT,\chi)-\frac{x^{\sigma+iT}}{\sigma+iT}\frac{L'}{L}(\sigma+iT,\chi)\right\}d\sigma$$

$$H_\chi^*(x,T,U)=\frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\left\{\frac{x^{\sigma-iT}}{\sigma-iT}\frac{L'}{L}(\sigma-iT,\chi)-\frac{x^{\sigma+iT}}{\sigma+iT}\frac{L'}{L}(\sigma+iT,\chi)\right\}d\sigma$$

V_χ and H_χ will be estimated by using lemma (xx) to bound $\frac{L'}{L}$.

Lemma 6.1. *If $|s+m|\ge 1/4$ for all non-negative integers m, then*

$$\left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right|\le\frac{n_M}{2}\left[\log\pi+3\gamma+8\log 2+\frac{1}{4}+5\log(|s|+3)\right]$$

Proof. This follows from applying lemma (5.5) to equation (5.7). □

Lemma 6.2. *If $\sigma \leq -1/4$ and $|s+m| \geq 1/4$ for all non-negative integers m, then*

$$\left|\frac{L'}{L}(s,\chi)\right| \leq \log A(\chi) + n_M\left[\log\pi + 3\gamma + 8\log 2 + 12\frac{1}{4} + 5\log\left(|s|+4\right)\right]$$

Proof. The functional equation (5.2) implies that:

$$\frac{L'}{L}(s,\chi) = -\frac{L'}{L}(1-s,\bar{\chi}) - \log A(\chi) - \frac{\gamma_\chi'}{\gamma_\chi}(1-s) - \frac{\gamma_\chi'}{\gamma_\chi}(s)$$

Since $\Re(1-s) \geq 5/4$ we can use corollaries (3.5) and (5.3) to bound $\left|\frac{L'}{L}(1-s,\bar{\chi})\right| \leq 12n_M$.

The previous lemma allows us to estimate the γ_χ terms. ∎

We now give crude (but sufficient) estimates for $V_\chi(x,T,U)$ and $H_\chi(x,T,U)$. We note that $U = j+1/2$ so $|-U+it+m| \geq 1/4$ for all integers m. Therefore

$$|V_\chi(x,T,U)| \leq \frac{1}{2\pi}\frac{x^{-U}}{U}\int_{+T}^{-T}\left|\frac{L'}{L}(-U+it,\chi)\right|dt \tag{6.2}$$

$$\leq \frac{1}{\pi}\frac{x^{-U}}{U}T\left[\log A(\chi) + 21n_M + 5n_M\log\left(U+T+4\right)\right]$$

and

$$|H_\chi(x,T,U)| \leq \frac{1}{2\pi}\int_{1/4}^{\infty}\frac{x^{-\sigma}}{T}2\left[\log A(\chi) + 21n_M + 5n_M\log\left(\sigma+4\right) + 5n_M\log T\right]d\sigma \tag{6.3}$$

$$\leq \frac{1}{\pi}\frac{x^{-1/4}}{T\log x}\left[\log A(\chi) + 21n_M + 5n_M\log T\right] + \frac{5n_M x^{-1/4}}{\pi T}\int_0^\infty x^{-\sigma}\log(\sigma+4\frac{1}{4})d\sigma$$

$$\leq \frac{1}{\pi}\frac{x^{-1/4}}{T\log x}\left[\log A(\chi) + 28.5n_M + 5n_M\log T\right]$$

(using $\int_0^\infty x^{-\sigma}\log(\sigma+4.25)d\sigma = \frac{1}{\log x}\int_0^\infty e^{-\sigma}\log(\frac{\sigma}{\log x}+4.25)d\sigma \leq \frac{1.5}{\log x}$ since $\log(\frac{\sigma}{10}+4.25) \leq 1.45+0.5\sigma$ and we assume $x \geq e^{10}$).

From lemma (5.11), we have:

$$\left|\frac{L'}{L}(\sigma+iT,\chi) - \sum_{|\gamma-T|\leq 1}\frac{1}{\sigma+iT-\rho}\right| \leq 2586n_M + 1510.5n_M\log(T+7) + 331.5\log A(\chi)$$

and a similar estimate hold at $\sigma - iT$. Therefore:

$$\left|H_\chi^*(x,T,U) - \frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\left\{\frac{x^{\sigma-iT}}{\sigma-iT}\sum_{|\gamma+T|\leq 1}\frac{1}{\sigma-iT-\rho} - \frac{x^{\sigma+iT}}{\sigma+iT}\sum_{|\gamma-T|\leq 1}\frac{1}{\sigma+iT-\rho}\right\}d\sigma\right| \leq$$

$$\leq 2\int_{-1/4}^{\sigma_0}\frac{x^\sigma}{T}\left\{2586n_M + 1510.5n_M\log(T+7) + 331.5\log A(\chi)\right\}d\sigma$$

$$\leq \frac{2ex}{T\log x}\left\{2586n_M + 1510.5n_M\log(T+7) + 331.5\log A(\chi)\right\}$$

We also need to show the first integral is not too large.

Lemma 6.3. *Let $\rho = \beta + i\gamma$ have $0 < \beta < 1$ and let $\gamma \neq t$. If $|t| \geq 2$, $x \geq 2$ and $1 < \sigma_1 \leq 3$ then*

$$\left|\int_{-1/4}^{\sigma_1}\frac{x^{\sigma+it}}{(\sigma+it)(\sigma+it-\rho)}d\sigma\right| \leq \frac{2x^{\sigma_1}}{(|t|-1)(\sigma_1-\beta)} + \frac{x^{\sigma_1}}{(|t|-1)}(\sigma_1+\frac{1}{4})$$

Proof. Suppose first that $\gamma > t$. Let B be the rectangle with vertices at $\sigma_1+i(t-1), \sigma_1+it, -\frac{1}{4}+it, -\frac{1}{4}+i(t-1)$ in this order. By Cauchy's theorem,

$$\oint_B \frac{x^s}{s(s-\rho)}ds = 0$$

since the integrand has no singularities inside the contour. However, on the two vertical sides of the rectangle, the integrand is majorized by $\frac{x^{\sigma_1}}{(|t|-1)(\sigma_1-\beta)}$ and on the lower horizontal side it is majorized by $\frac{x^{\sigma_1}}{|t|-1}$. In the case $\gamma < t$ one uses the rectangle with vertices at $\sigma_1+i(t+1), \sigma_1+it, -\frac{1}{4}+it, -\frac{1}{4}+i(t+1)$. □

The above lemma shows that:

$$\left|\frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\left\{\frac{x^{\sigma-iT}}{\sigma-iT}\sum_{|\gamma+T|\le 1}\frac{1}{\sigma-iT-\rho}-\frac{x^{\sigma+iT}}{\sigma+iT}\sum_{|\gamma-T|\le 1}\frac{1}{\sigma+iT-\rho}\right\}d\sigma\right| \le \frac{1}{2\pi}\left(\frac{2}{\sigma_0-1/2}+\sigma_0+\frac{1}{4}\right)\frac{x^{\sigma_0}}{T-1}\left(n_\chi(-T)+n_\chi(T)\right)$$

where we used GRH to assume $\beta = 0.5$ for all the roots. We now assume $x \ge e^{10}$ so $\sigma_0 \le 1.1$ and $T \ge e^5$. Using lemma (5.7), we get:

(6.4)

$$\left|H_\chi^*(x,T,U)\right| \le \frac{2ex}{T\log x}\{2585n_M+1510.5n_M\log(T+7)+331.5\log A(\chi)\}+\frac{8.6ex}{T}\{9n_M+3n_M\log(T+1)+\log A(\chi)\}$$

We can now combine equations (6.2), (6.3) and (6.4) to get:

$$(6.5)\qquad |I_\chi(x,T)-I_\chi(x,T,U)| \le |V_\chi(x,T,U)|+|H_\chi(x,T,U)|+\left|H_\chi^*(x,T,U)\right|$$

$$\le \frac{1}{\pi}\frac{Tx^{-U}}{U}\left[\log A(\chi)+21n_M+5n_M\log\left(U+T+4\right)\right]+\frac{1}{\pi}\frac{x^{-1/4}}{T\log x}\left[\log A(\chi)+21n_M+5n_M\log T\right]$$

$$+\frac{2ex}{T\log x}\{2586n_M+1510.5n_M\log(T+7)+331.5\log A(\chi)\}+\frac{8.6ex}{T}\{9n_M+3n_M\log(T+1)+\log A(\chi)\}$$

7. The explicit formula

In this section we combine the results of the preceding sections in order to obtain an explicit formula for $\psi_C(x)$ in terms of the zeroes ρ.

We first evaluate the contour integral $I_\chi(x,T,U)$ defined by (6.1). We assume that $x \ge e^{10}$, $U = j - 1/2$ for some non-negative integer j and $T \ge 2$ doesn't equal the ordinate of any zero of the $L(s,\chi)$. By the residuum theorem, $I_\chi(x,T,U)$ equals the sum of the residues of the integrand at the poles inside $B_{T,U}$. Now, if $\chi = \chi_1$, the principal character, then L'/L has a first-order pole of residue -1 at $s = 1$, and hence (this term being absent for $\chi \ne \chi_1$), we obtain a contribution of

$$-\delta(\chi)x$$

from this possible pole. Further, L'/L has a first-order pole with residue $+1$ at each zero $L(s,\chi)$ (counted according to multiplicity). The nontrivial zeros contribute together:

$$\sum_{|\gamma|<T}\frac{x^\rho}{\rho}$$

In addition, the functional equation, together with the definition of the completed L-function, shows that L'/L has first order poles at $s = -(2m-1), m = 1,2,\ldots$, where the residue is $b(\chi)$, and first order poles at $s = -2m, m = 0,1,2,\ldots$, where the residue is $a(\chi)$. Hence, the residues at points with $\Re(s) < 0$ contribute:

$$-b(\chi)\sum_{m=1}^{\left[\frac{U+1}{2}\right]}\frac{x^{-(2m-1)}}{2m-1}-a(\chi)\sum_{m=1}^{\left[\frac{U}{2}\right]}\frac{x^{-2m}}{2m}$$

The remaining residue is at $s = 0$, where both x^s/s and L'/L may have first-order poles. We write the Laurent expansions of the two functions as follows:

$$\frac{x^s}{s}=\frac{1}{s}+\log x+O(s)$$

and (using (5.3))

$$\frac{L'}{L}(s,\chi)=\frac{a(\chi)-\delta(\chi)}{s}+r(\chi)+O(s)$$

where

$$r(\chi)=B(\chi)-\frac{1}{2}\log A(\chi)+\frac{n_M}{2}\log\pi+\delta(\chi)-\frac{b(\chi)}{2}F(\frac{1}{2})-\frac{a(\chi)}{2}F(1)$$

Hence the residue at $s = 0$ is

$$r(\chi)+(a(\chi)-\delta(\chi))\log x$$

If we now collect all these residue terms, we find that:

$$I_\chi(x,T,U) = -\delta(\chi)x + \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - b(\chi) \sum_{m=1}^{\left[\frac{U+1}{2}\right]} \frac{x^{-(2m-1)}}{2m-1} - a(\chi) \sum_{m=1}^{\left[\frac{U}{2}\right]} \frac{x^{-2m}}{2m} + r(\chi) + (a(\chi)-\delta(\chi))\log x$$

Letting $U \to \infty$ and using the estimate (6.6) we obtain:

$$\left| I_\chi(x,T) + \delta(\chi)x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - r(\chi) - (a(\chi)-\delta(\chi))\log x - \frac{n_M}{2}\log(1-x^{-1}) + \frac{1}{2}(b(\chi)-a(\chi))\log(1+x^{-1}) \right|$$

$$\begin{aligned} \leq \quad & \frac{1}{\pi}\frac{x^{-1/4}}{T\log x}\left[\log A(\chi) + 28.5n_M + 5n_M\log T\right] + \\ & + \frac{2ex}{T\log x}\left\{2586n_M + 1510.5n_M\log(T+7) + 331.5\log A(\chi)\right\} + \frac{8.6ex}{T}\left\{9n_M + 3n_M\log(T+1) + \log A(\chi)\right\} \end{aligned}$$

$$\leq \frac{x}{T}\left\{351n_M + 375.2n_M\log T + 196.6\log A(\chi)\right\} \tag{7.1}$$

where we assumed $x \geq e^{10}, T \geq e^5$.

Note that letting $T \to \infty$ would give and explicit formula for the inverse Mellin transform

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{x^s}{s}\frac{L'}{L}(s,\chi)ds$$

with no error term. However, for our purposes the cruder version is more useful.

Theorem 7.1. *If $x \geq e^{10}$ and $T \geq e^5$ then*

$$\begin{aligned} \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x,T)\right| \quad \leq \quad & \frac{|C|}{|G|}\frac{196.6x + 32.5T}{T}\log d_L + \frac{|C|}{|G|}\left\{\frac{x}{T}(351 + 375.2\log T) + 350.6 + \log x\right\} n_L \\ + \quad & 2\log x\log d_L + 2.51n_K\log x + 5n_K x T^{-1}\log^2 x + 36.55n_K T^{-1} x\log x \end{aligned}$$

where

$$S(x,T) = \frac{|C|}{|G|}\sum_\chi \bar\chi(g) \sum_{|\gamma|<T} \frac{x^\rho}{\rho}$$

[The inner sum is over the nontrivial zeros of $L(s,\chi)$]

Proof. Corollary (5.9) shows that:

$$|B(\chi)| \leq 348.36n_M + 32\log A(\chi)$$

and therefore (using $a(\chi)+b(\chi) = n_M$, $F(1) = -\gamma$, $F(\frac{1}{2}) = -\gamma - \log 2$)

$$r(\chi) = 348.36n_M + 32.5\log A(\chi) + \frac{n_M}{2}\log\pi + \delta(\chi) - \frac{b(\chi)}{2}F(\frac{1}{2}) - \frac{a(\chi)}{2}F(1)$$

$$|r(\chi)| \leq 350.6n_M + 32.5\log A(\chi)$$

And so

$$\left| I_\chi(x,T) + \delta(\chi)x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right| \leq \frac{x}{T}\left\{351n_M + 375.2n_M\log T + 196.6\log A(\chi)\right\} + 350.6n_M + 32.5\log A(\chi) + n_M\log x$$

Summing over the characters, we then find:

$$\begin{aligned} \left|\frac{|G|}{|C|}I_C(x,T) - \sum_\chi \bar\chi(g)\left\{\delta(\chi)x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right\}\right| \quad \leq \quad & \frac{196.6x + 32.5T}{T}\sum_\chi \log A(\chi) + [L:M]n_M \cdot \\ \cdot \quad & \left\{\frac{x}{T}(351 + 375.2\log T) + 350.6 + \log x\right\} \end{aligned}$$

We then note that $\sum_\chi \log A(\chi) = \log d_L$ by the conductor-discriminant formula. Also, $n_M \cdot [L:M] = n_L$ and $\sum_\chi \bar\chi(g)\delta(\chi) = 1$. Thus:

$$\left|\frac{|G|}{|C|}I_C(x,T) - x + \sum_\chi \bar\chi(g)\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right| \leq \frac{196.6x+32.5T}{T}\log d_L + \left\{\frac{x}{T}(351+375.2\log T)+350.6+\log x\right\}n_L$$

Which shows $I_C(x,T)$ as the main term $\frac{|C|}{|G|}x$ with an estimate on the remainder. Using the estimate (3.14) gives the final result. ∎

8. Zero-free regions

We make the strongest possible assumption here (GRH) and skip this section.

9. Final Estimates

Theorem 9.1. *If $\zeta_L(s)$ satisfies the GRH and $x \geq e^{10}$:*

$$\left|\psi_C(x) - \frac{|C|}{|G|}x\right| \leq \frac{|C|}{|G|}44x^{1/2}\log x\log d_L + \left(12.5\frac{\log x}{|G|} + 337\frac{|C|}{|G|}\right)x^{1/2}\log x n_L + 2\log x\log d_L$$

Proof. We need to estimate $S(x,T)$. Since $|x^\rho| = x^{1/2}$, we have:

$$x^{-1/2}\left|\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right| \leq \sum_{|\gamma|<T}\frac{1}{|\rho|} \leq \sum_{j=1}^{[T]}\frac{n_\chi(j)}{j} + 2n_\chi(0) + \frac{n_\chi(|T|)}{[T]}$$

(each positive zero is counted twice, so there's no need to consider negative zeros). We now use the estimate (5.7) and get:

$$\left|\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right| \leq 5x^{1/2}\left\{2\log A(\chi) + 18n_M + \left(\sum_{j=1}^{[T]}\frac{1}{j} + \frac{1}{T-1}\right)(\log A(\chi) + 3n_M\left[3+\log(T+1)\right])\right\}.$$

Assuming $T \geq e^5$ this reads:

$$\left|\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right| \leq x^{1/2}\log T\left\{8\log A(\chi) + 77.1n_M + 3.1\log T\right\}.$$

and summing over the characters,

$$|S(x,T)| \leq \frac{|C|}{|G|}x^{1/2}\log T\left\{8\log d_L + 77.1n_L + 3.1\log T\right\}$$

which can be combined with the theorem to give

$$\begin{aligned}\left|\psi_C(x) - \frac{|C|}{|G|}x\right| \quad &\leq \quad \frac{|C|}{|G|}\frac{196.6x+32.5T}{T}\log d_L + \frac{|C|}{|G|}\left\{\frac{x}{T}(351+375.2\log T)+350.6+\log x\right\}n_L \\ &+ \quad \frac{|C|}{|G|}x^{1/2}\log T\left\{8\log d_L + 77.1n_L + 3.1\log T\right\} + 2\log x\log d_L + 2.51n_K\log x + 5n_KT^{-1}x\log^2 x + 36.6n_KT^{-1}x\log x\end{aligned}$$

We now set $T = \sqrt{x}$ and get:

$$\left|\psi_C(x) - \frac{|C|}{|G|}x\right| \leq \frac{|C|}{|G|}44x^{1/2}\log x\log d_L + \left(12.5\frac{\log x}{|G|} + 337\frac{|C|}{|G|}\right)x^{1/2}\log x n_L + 2\log x\log d_L$$

where we note that $|G| = [L:K] = n_L/n_K$. ∎

Lemma 9.2. *(Summation by parts) If $\left|\psi_C(x) - \frac{|C|}{|G|}x\right| \leq g(x)$ for $x \geq x_0$ then*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\operatorname{Li}(x)\right| \leq \frac{g(x)}{\log x} + \int_{x_0}^x \frac{g(t)dt}{t\log^2 t} + x^{1/2} + x^{1/3}\log x + \operatorname{Li}(x_0) + \frac{2}{\log 2}$$

Proof. We consider first the set

$$\tilde{P}_C = \{\mathfrak{p}^m |\ \mathfrak{p} \text{ unramified in } L, \left[\frac{L/K}{\mathfrak{p}}\right] = C\}$$

and its characteristic function (also denoted $\tilde{P}_C$). In this notation we have:

$$\psi_C(x) = \sum_{n\le x} \tilde{P}_C(n)\log n$$

and we also set

$$\tilde{\pi}_C(x) = \left|\tilde{P}_C \cap (1,x)\right| = \tilde{\pi}_C(x) = \sum_{n\le x} \tilde{P}_C(n) = \sum_{n\le x} \frac{\tilde{P}_C(n)\log n}{\log n}.$$

Since

$$\frac{1}{\log n} - \frac{1}{\log x} = \int_n^x \frac{dt}{t\log^2 t}$$

we have:

$$\tilde{\pi}_C(x) = \sum_{n\le x} \tilde{P}_C(n)\log n \int_n^x \frac{dt}{t\log^2 t} + \frac{1}{\log x}\sum_{n\le x} P_C(n)\log n$$

interchanging the summation and the integration gives:

$$\tilde{\pi}_C(x) = \int_2^x \frac{\psi_C(t)}{t\log^2 t}dt + \frac{\psi_C(x)}{\log x}.$$

Integration by parts gives:

$$\int_2^x t\cdot\frac{1}{t\log^2 t}dt = \frac{2}{\log 2} - \frac{x}{\log x} + \mathrm{Li}(x).$$

So we finally have:

$$\left|\tilde{\pi}_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \le \int_2^x \frac{dt}{t\log^2 t}\left|\psi_C(t) - \frac{|C|}{|G|}t\right| + \frac{1}{\log x}\left|\psi_C(x) - \frac{|C|}{|G|}x\right| + \frac{2}{\log 2}$$

or

$$\left|\tilde{\pi}_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \le \frac{g(x)}{\log x} + \int_{x_0}^x \frac{g(t)dt}{t\log^2 t} + \mathrm{Li}(x_0) + \frac{2}{\log 2}.$$

We also need to bound the difference $|\tilde{\pi}_C(x) - \pi_C(x)|$, i.e. the terms coming from prime powers. However, this is clearly at most

$$x^{1/2} + x^{1/3} + x^{1/4} + \ldots \le x^{1/2} + x^{1/3}\log x.$$

□

We can now prove the main result:

Theorem 9.3. *(Čebotarev Density Theorem) For $x \ge e^{10}$ we have:*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \le \frac{|C|}{|G|}44x^{1/2}\log d_L + \left(12.5\frac{\log x}{|G|} + 337\frac{|C|}{|G|}\right)x^{1/2}n_L + 2\log d_L.$$

Proof. The estimate (valid for $x \ge x_0 = e^{10}$)

$$g(x) = \frac{|C|}{|G|}44x^{1/2}\log x\log d_L + \left(12.5\frac{\log x}{|G|} + 337\frac{|C|}{|G|}\right)x^{1/2}\log x n_L + 2\log x\log d_L$$

of theorem 9.1 together with the lemma implies:

$$\begin{aligned}\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| &\le \frac{|C|}{|G|}44x^{1/2}\log d_L + \left(12.5\frac{\log x}{|G|} + 337\frac{|C|}{|G|}\right)x^{1/2}n_L + 2\log d_L + \\ &+ \int_{x_0}^x \frac{g(t)dt}{t\log^2 t} + x^{1/2} + x^{1/3}\log x\end{aligned}$$

Where I'm sure that $\mathrm{Li}(x_0)+\frac{2}{\log 2}\approx 42$ term is negligible compared to the generosity of the other estimates. Now:

$$\int_{x_0}^{x}\frac{t^{1/2}dt}{t\log t}\leq x^{1/2}\log\log x$$

and

$$\int_{x_0}^{x}\frac{t^{1/2}dt}{t\log^2 t}\leq\frac{x^{1/2}}{\log x}$$

and in both cases we have been sufficiently generous with the constants elsewhere to ignore these terms. Finally for $x\geq x_0$, $x^{1/2}+x^{1/3}\log x\leq 3x^{1/2}$ which can be neglected as well. $\square$

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