This examination has 15 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

7 June 2007, 12:00-15:00

Name: ______________________ **Signature:** ______________________

School: ______________________ **Candidate Number:** ______________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n-1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		4
2		6
3		6
4		6
5		6
6		6
7		8
8		7
9		5
10		8
11		10
12		6
13		8
14		8
15		6
Total		100

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Formula Sheet for 7 June 2007

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

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[4] **1.** Find an equation for the line tangent to the following curve at the point where $x = 1$:

$$y = \frac{6x - 2/x}{x^2 + \sqrt{x}}.$$

[6] **2.** Find an equation for the line tangent to the following curve at the point $(2, 0)$:

$$y = 2 + x - x^2 - \sin(xy).$$

Continued on page 4

[6] **3.** Find the derivatives of the three functions below. Do not simplify.

$$a(x) = \tan\left(x + e^{-x}\right)$$

$$b(x) = e^{x^2} \tan^{-1}\left(x^3 - x\right)$$

$$c(x) = \frac{\sin^{-1}(x)}{\sin^{-1}(2x)}$$

Continued on page 5

[6] **4.** For each limit below, find the exact value (with justification) or explain why the limit does not exist.

(a) $\displaystyle\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x}\right)$.

(b) $\displaystyle\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{6}{x^2-9}\right)$.

[6] **5.** Use the definition of the derivative as a limit to find $f'(x)$, given

$$f(x)=\frac{x}{1+x}.$$

(Finding $f'(x)$ using differentiation rules will earn no marks, but it could help you check your work with limits.)

Continued on page 6

[6] **6.** Let $f(t) = \dfrac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}}$ where $r > 0$ is a constant.

(a) Derive the "small-t approximation"

$$f(t) \approx rt \qquad \text{for } t \approx 0.$$

(b) Explain why the approximate value rt is greater than the exact value $f(t)$ whenever $t > 0$. What happens when $t < 0$?

Continued on page 7

[8] **7.** A conical reservoir with an open top and vertex down holds water for a desert community. The reservoir is 6 metres deep at its centre; its diameter at the top is 10 metres. The rate at which water evaporates from the reservoir is proportional to the area of the water's top surface (a circular disk). When the water is 5 metres deep, its depth is decreasing at the instantaneous rate of 2 cm per day. Find, with suitable units, ...

(a) the rate of change of depth when there is 3 metres of water in the reservoir, and

(b) the rate of change of volume when there is 3 metres of water in the reservoir.

Continued on page 8

[7] **8.** The position of a moving particle at time t is $x = s(t)$, where

$$s(t) = \ln(1+t) - \frac{t}{1+t}.$$

(a) Express the particle's velocity as a function of t. Simplify your answer.

(b) Explain why $s(t) > 0$ whenever $t > 0$.

(c) Find the particle's maximum velocity, and the time when it occurs.

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[5] **9.** Let $f(x) = \left(1 + \frac{3}{x}\right)^x$.

By using your findings from the previous question, or otherwise, ...

(a) Show that f is increasing on the interval $x > 0$.

(b) Find the exact value of $\lim_{x\to\infty} f(x)$.

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[8] **10.** A cup of coffee at 96°C is set on a table in an air-conditioned classroom. It cools to 60°C in 10 minutes, and then to 40°C in another 10 minutes. What is the temperature of the room?

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[10] **11.** Let $f(x) = x^2\sqrt{24 - x^2}$.

(i) Find the domain of f.

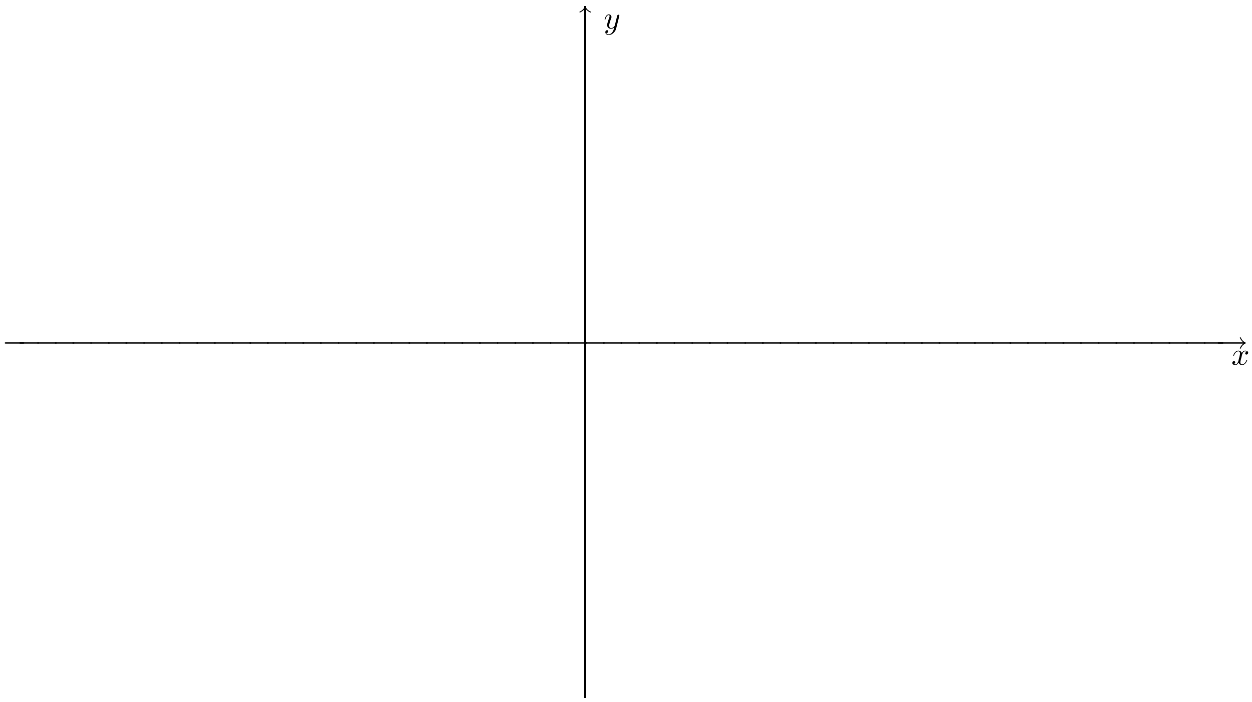
(ii) Find the intervals in which f is increasing and decreasing.

(iii) Find the absolute maximum and minimum values for f on its domain, and all the points where these are attained.

(iv) Find the x-coordinates of all inflection points for f.

(v) Sketch the graph of f. (A pair of axes is supplied on the next page.)

Continued on page 12



Continued on page 13

[6] **12.** The nonlinear equation $x^{-1}\sin(x) = \cos(x)$ has a solution near the point $x_0 = 3\pi/2$. Use the tangent lines at $x = x_0$ to the two curves

$$y = x^{-1}\sin(x), \qquad y = \cos(x)$$

to find a better approximation (call it x_1) to the solution near x_0.

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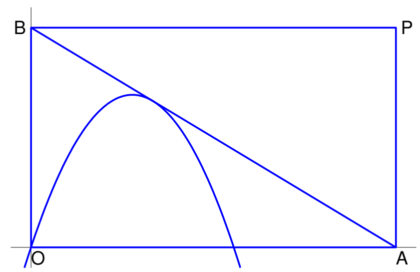
[8] **13.** Every tangent line of negative slope for the curve $y = 3x - x^2$ can be used to construct a rectangle, as shown below: put one corner at the origin (O), one at the line's x-intercept (A), one at the line's y-intercept (B), and one in the first quadrant to complete the figure (P). Find the coordinates of P for the rectangle of *smallest perimeter* that can be constructed this way. (The sketch shows the construction, but not the minimizing configuration.)



Continued on page 15

[8] **14.** Use the three properties below to identify the function f and sketch its graph:

(i) $f(0) = 0$, and

(ii) the graph of f has an inflection point at which the tangent line is horizontal, and

(iii) $f''(x) = 6x + 2$ for all x.

[6] **15.** Make a rough sketch of the following figures on the same set of axes:

$$C: \quad y = \frac{10}{1+x^2}, \qquad L: \quad y = 5.$$

Then find the area of the region lying below C and above L.

The End