Calculus Exam 2007 — Commentary

[4] **1.** Most writers got the idea. The number of terms involved led to many errors in calculation.

[6] **2.** About half the students who didn't succeed in this question got into trouble with the chain-rule based step

$$\frac{d}{dx}\sin(xy) = \cos(xy)\,[y + xy'] \,.$$

[6] **3.** Almost all writers found $a'(x)$ correctly. A few forgot the chain-rule factor $(3x^2 - 1)$ in the second term of $b'(x)$. A significant number missed the factor of 2 in the second numerator term of $c'(x)$. Predictably, some students misinterpreted $\sin^{-1}(x)$ as $1/\sin(x)$ and/or $\tan^{-1}(x)$ as $1/\tan(x)$.

[6] **4.** Only a handful of students received full marks in part (a). Recognizing the limiting value as 2, especially with the aid of a calculator, is not too difficult . . . but not many writers supplied the requested justification.

[6] **5.** Correct use of limit notation was required for full credit. Constellations of symbols that cost students marks included

$$\lim_{h\to 0} = \frac{f(x+h) - f(x)}{h}.$$

Likewise, the equation

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{\left(\frac{x+h}{1+x+h}\right) - \left(\frac{x}{1+x}\right)}{h}$$

is not correct: the left side is a number that depends on x, while the right side is an expression that depends on both x and h.

[6] **6.** Only a handful of students earned 5 or 6 points here. Finding $f'(t)$ and making a linear approximation earned 1 or 2 points; recognizing that this completed part (a) gave a third mark. Simply mentioning concavity earned one mark toward part (b); two more were reserved for handling the details.

[8] **7.** Students received 2 marks for finding similar triangles for general values of radius r and depth h, 2 more for the evaporation rate equation, another 2 for a relevant related-rates idea, and 1 each for getting the answers for parts (a) and (b).

[7] **8.** This was generally well done.

[5] **9.** This was given a low point value because the examiners expected it to be difficult. In part (a), logarithmic differentiation was attempted by about half the students; most of these succeeded (for 2 marks). Justifying $f'(x) > 0$ was rare: a few students

managed to make a connection with Question 8; others studied $f'(x)$ directly, showing that it has no positive roots and satisfies $f'(a) > 0$ at some point $a > 0$. Only a handful sorted out the limit in (b) properly. Some solvers made what amounts to this calculation based on substituting $t = x/3$:

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \left[\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x/3}\right]^3 = \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right]^3 = [e]^3.$$

(The final equation requires recalling a standard fact, perhaps from the theory of continuously compounded interest.)

[8] **10.** About half the students left this blank. Among those who wrote something and seemed to understand the point of the question, there was an even split between those who got stuck solving the system of 2 equations in 2 unknowns and those who completed the solution correctly.

[10] **11.** A nice trick contributed by one student:

$$f(x)^2 = 24x^4 - x^6 \implies 2f(x)f'(x) = 96x^3 - 6x^5 \implies f'(x) = \frac{3x(16-x^2)}{\sqrt{24-x^2}}.$$

Factoring f' is very helpful in finding intervals of increase and decrease. The expanded form is best for finding f'', but factoring the latter helps with a concavity analysis.

[6] **12.** Simply writing down an accurate numerical approximation to the intersection point (presumably taken from the calculator) earned no marks at all. On the other hand, each tangent line equation was worth 2 marks, and the x-coordinate of their intersection point was worth another 2. This setup is logically equivalent to taking one step Newton's root-finding method: students who defined $h(x) = x\cos(x) - \sin(x)$ or $k(x) = \cos(x) - \frac{\sin(x)}{x}$ and correctly took one Newton step from the initial point x_0 arrived at the same point x_1 and received full credit.

[8] **13.** The hard part of this problem is finding what function is to be minimized. Half of the 8 marks available were given for success in identifying f; identifying a critical point in the relevant domain earned 2 more; analyzing the critical points and returning the coordinates of P as requested triggered the release of another 2 points.

[8] **14.** This was generally well done.

[6] **15.** Many writers succeeded here.