# SFU – UBC – UNBC – Uvic
# Calculus Challenge Examination
# June 5, 2008, 12:00 – 15:00

**Host: SIMON FRASER UNIVERSITY**

**First Name:** ______________________

**Last Name:** ______________________

**School:** ______________________

Student signature

**INSTRUCTIONS**

1. Show all your work. Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete.
2. Calculators are optional, not required. Correct answer that is calculator ready, like $3+\ln 7$ or $e^2$, are preferred.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page *n*, work on the back of the page *n*-1.
6. CAUTION – Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.

| Question | Maximum | Score |
|---|---|---|
| 1 | 9 | |
| 2 | 6 | |
| 3 | 6 | |
| 4 | 6 | |
| 5 | 8 | |
| 6 | 5 | |
| 7 | 6 | |
| 8 | 6 | |
| 9 | 6 | |
| 10 | 8 | |
| 11 | 8 | |
| 12 | 9 | |
| 13 | 9 | |
| 14 | 8 | |
| Total | 100 | |

# Formula Sheet for 5 June 2008

## Exact Values of Trigonometric Functions

| $\theta$ | $0$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $3\pi/4$ | $5\pi/6$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $1/2$ | $\sqrt{2}/2$ | $\sqrt{3}/2$ | $1$ | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | $0$ |
| $\cos\theta$ | $1$ | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | $0$ | $-1/2$ | $-\sqrt{2}/2$ | $-\sqrt{3}/2$ | $-1$ |

## Trigonometric Definitions and Identities

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

[9] 1. In each case either compute the limit explaining briefly how you obtained the value, or explain why the limit does not exist.

(a) $\lim_{x\to 0} \frac{3}{4^{1/x}+1}$

(b) $\lim_{x\to -\infty} \left( \frac{x^4-5}{x^3+2x^2} - \frac{x^5+1}{x^4-1} \right)$

(c) $\lim_{x\to \infty} \left(1+\frac{3}{x}\right)^{2x}$

[6] 2. Find the derivatives of the functions below. Do not simplify.

(a) $g(x) = \sec\left(5x^2 - \tan(2x)\right)$

(b) $h(x) = 5\left(\sqrt[3]{x} + 1\right)e^{x^2}$

[6] 3. Given $x^{\cos y} = y^{\sin x}$ use logarithmic differentiation to find an expression for $\frac{dy}{dx}$ in terms of $x$ and $y$. No need to simplify the expression.

[6] 4. The limit $\lim_{h\to 0}\dfrac{\sqrt{9-2h}-3}{h}$ represents the derivative of some function *f* at the point $x=0$.

(a) Find one possible function definition for *f* such that $f'(0)=\lim_{h\to 0}\dfrac{\sqrt{9-2h}-3}{h}$.

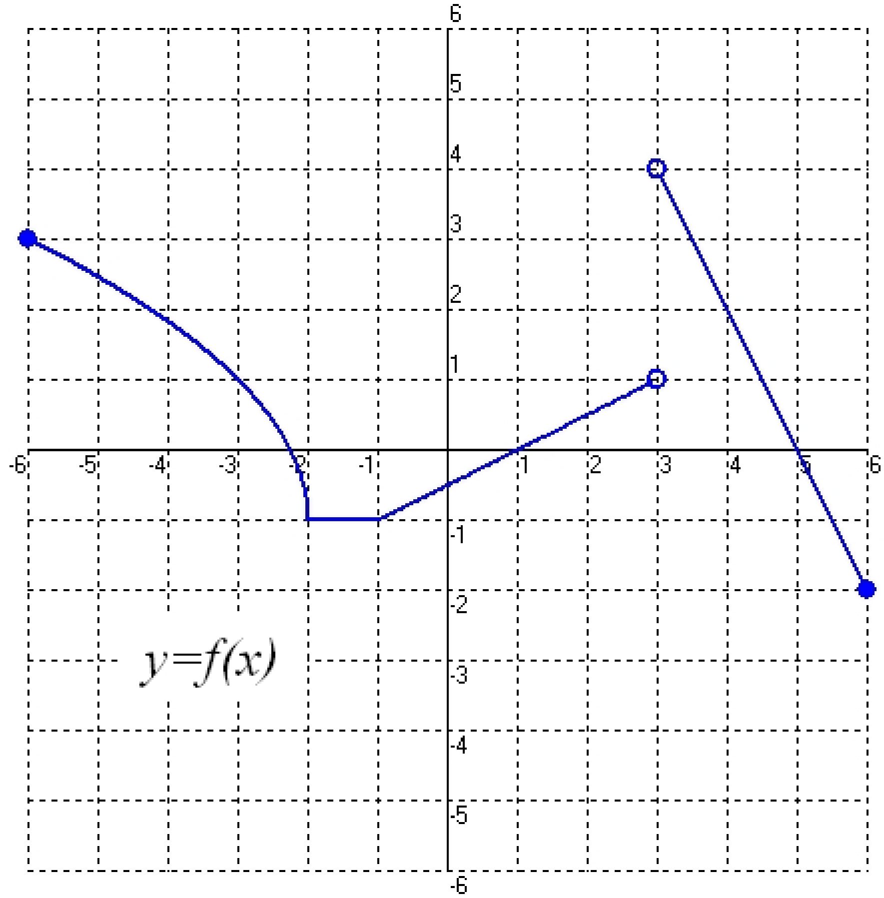
(b) Evaluate the limit directly without using the fact that it is equal to $f'(0)$.

[8] 5. What is an equation for the straight line through the point $(3,0)$ that is tangent to the graph of $y = x + \frac{3}{x}$ at a point in the first quadrant?

[5] 6. Recall the definition of the inverse tangent function: $\theta = \tan^{-1} t \iff t = \tan\theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Show that $\frac{d}{dt}\left(\tan^{-1} t\right) = \frac{1}{1+t^2}$.

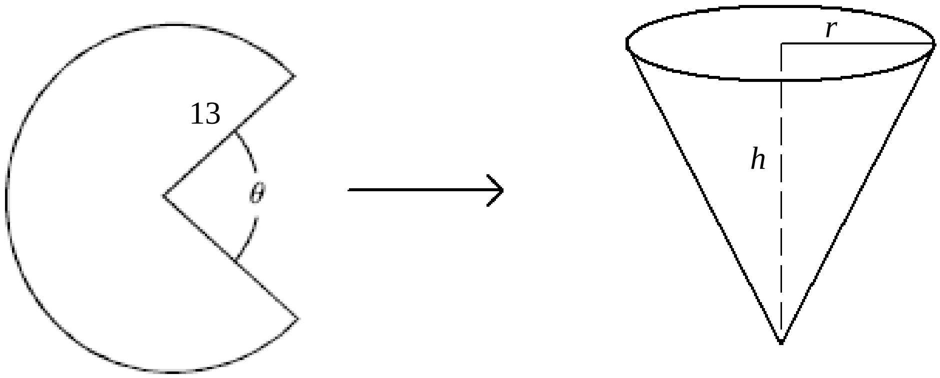
[6] 7. Consider the graph of the function $y = f(x)$ shown below. Estimate the slope of the graph of $f$ at various points and use these estimates to sketch the graph of $y = f'(x)$.



[6] 8. Use linear approximation (or differentials) to estimate $(1.99)^4$.

[6] 9. Find the largest interval on which the graph of the function $f(x) = \dfrac{\ln x}{x}$ is concave up.

[8] 10. A cone is to be constructed from a circular piece of paper with radius 13 centimetres by cutting out a wedge as shown in the diagram below. What is the maximum volume of the cone? (Hint: $V_{cone} = \frac{1}{3}\pi r^2 h$)



[8] 11. A particle is moving along the curve $f(x) = x^2$. As the particle passes through the point $(3, f(3))$, its $x$-coordinate increases at a rate of 5 cm/s. How fast is the distance from the particle to the point $(0, f(0))$ changing at this instant?

[9] 12. Sketch the graph of a function $f$ with the following properties.

(a) $f$ is continuous on its domain $\{x \in \mathbb{R} / x \neq -3, 1\}$.

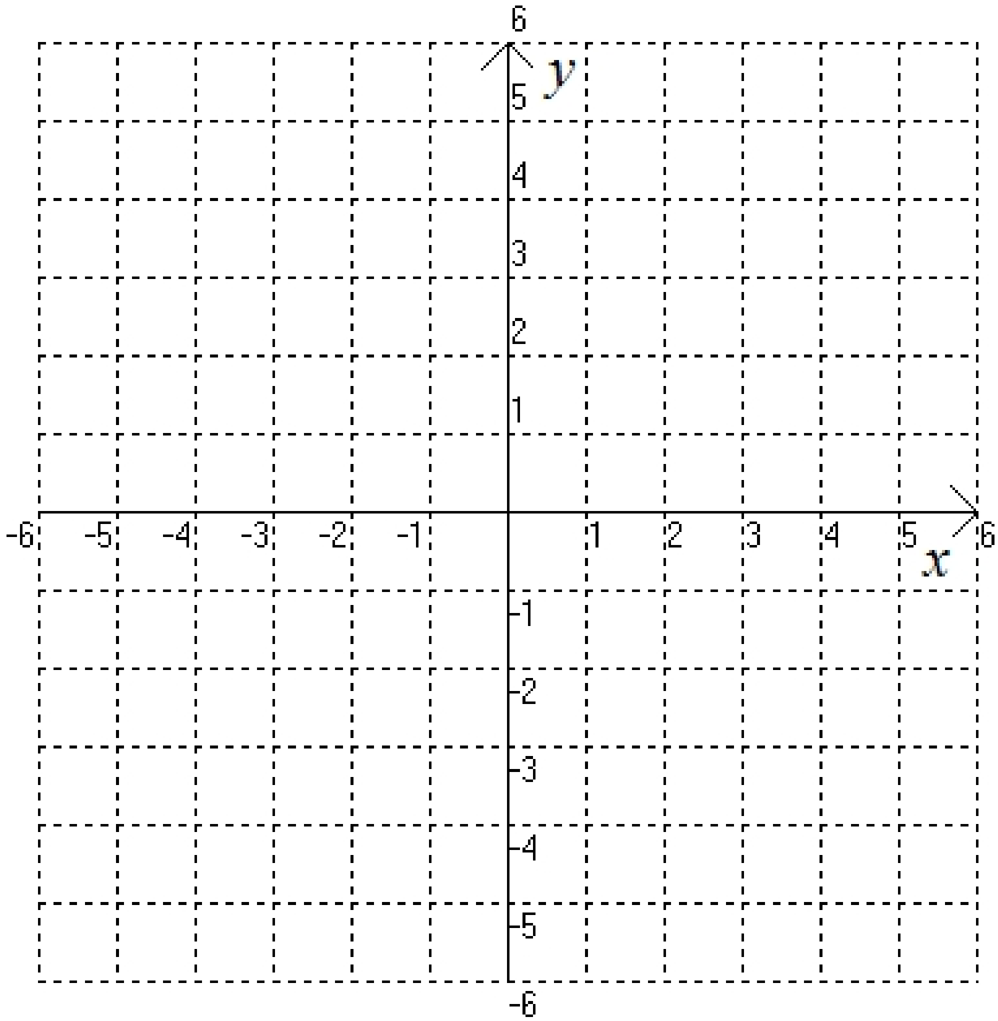
(b) $f(0) = 2$ and $f(4) = 1$ are inflection points.

(c) $f(3) = 4$, $f'(3) = 0$, and $f''(3) < 0$.

(d) $\lim_{x \to \infty} f(x) = -2$ and $\lim_{x \to -\infty} f(x) = -2$.

(e) $\lim_{x \to -3^+} f(x) = \infty$, $\lim_{x \to -3^-} f(x) = -\infty$, $\lim_{x \to 1^+} f(x) = -\infty$ and $\lim_{x \to 1^-} f(x) = -\infty$.

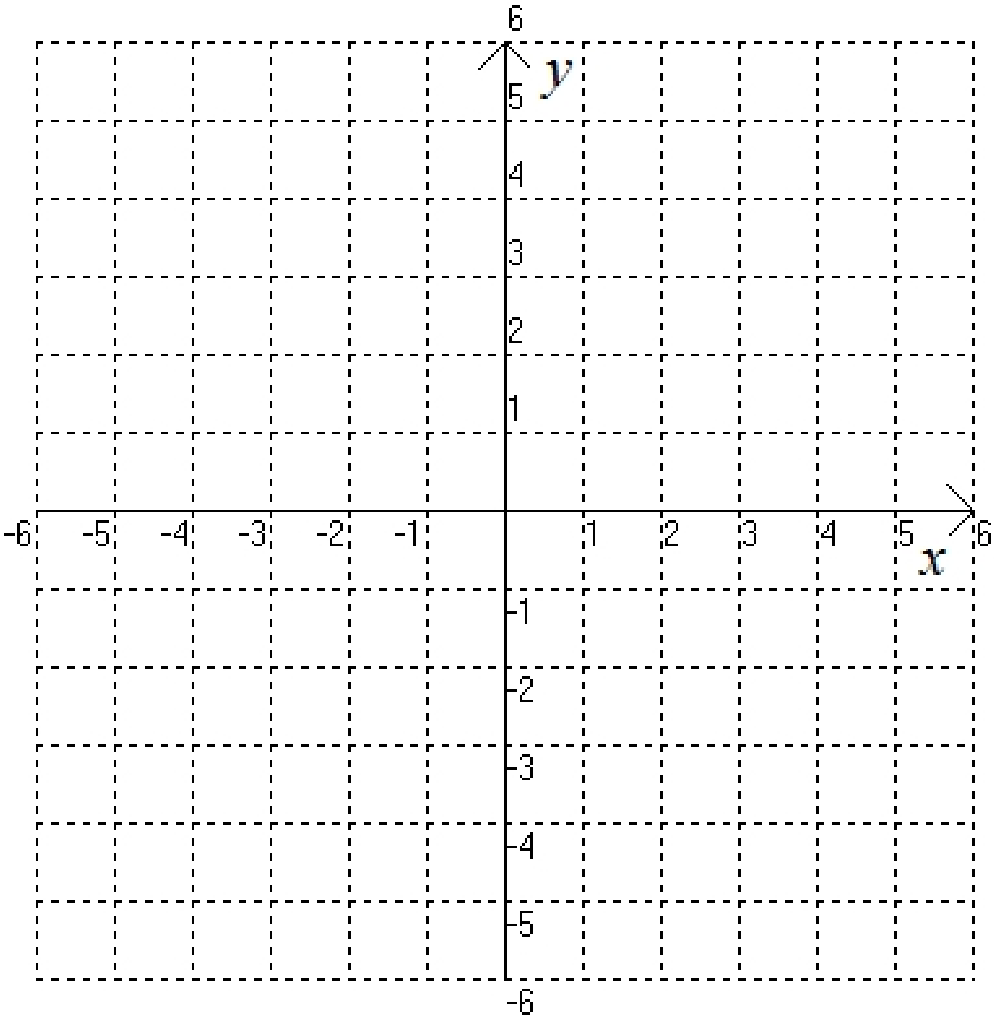
(f) $f'(x) < 0$ for $x < -3$, $-3 < x < 1$, and $x > 4$ (should be $x > 3$), and $f'(x) > 0$ for $1 < x < 4$ (should be $1 < x < 3$).




(next page 15 left blank for work on question 12)

(page 15 for additional work for question 12 if needed)

The grid below is provided for rough work only. Show your final work on page 14.



[9] 13. The levels of a sedative in a patient's blood were monitored to determine the appropriate time for an operation. Every fifteen minutes a blood sample was taken to determine the concentration $C$ of the sedative in milligrams per litre, and then recorded in the table of data shown below.

| Time (min) | Concentration $C$ (mg/l) |
|---|---|
| 0 | 20 |
| 15 | 10.21 |
| 30 | 5.15 |
| 45 | 2.68 |
| 60 | 1.31 |
| 75 | 0.72 |

(a) Estimate the rate of change of concentration with respect to time at 30 minutes and 60 minutes. Is the rate of change of concentration with respect to time $t$ a constant?

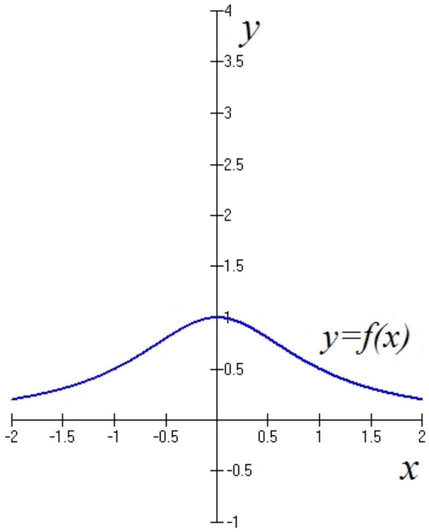
(b) Show that the rate of change is roughly proportional to the concentration. Write this relationship as a differential equation leaving the constant of proportionality, $k$, undetermined.

(c) Solve the differential equation from part (b) and choose the constant of proportionality, $k$, so that the solution satisfies both the entries $C(0) = 20$ and $C(60) = 1.31$ from the table. Write the constant of proportionality accurate to 4 decimal places.

(next page 17 left blank for work on question 13)

(page 17 for additional work for question 13 if needed)

[8] 14. Below is the graph of $f(x) = \dfrac{1}{1+x^2}$.




(a) Graph and shade the region enclosed by the curves $x = \pm\sqrt{3}$, $y = \pi$, and $f(x) = \dfrac{1}{1+x^2}$.

(b) Find the area of the region described in part (a). (Hint: You may use information from a previous question on this exam.)