This examination has 15 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

4 June 2009, 12:00-15:00 PDT

Name: ______________________ **Signature:** ______________________

School: ______________________ **Candidate Number:** ______________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. Some basic formulas appear on page 2. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n-1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		6
2		4
3		5
4		7
5		6
6		6
7		8
8		9
9		6
10		9
11		9
12		9
13		9
14		7
Total		100

UBC-SFU-UVic-UNBC Calculus Examination

Formula Sheet for 4 June 2009

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

Continued on page 3

[6] **1.** Find the derivative of each function below. Do not simplify.

(a) $f(x) = \dfrac{\sin(5x)}{1+x^2}$

$$f'(x) = \frac{(5\cos(5x))\,[1+x^2] - [\sin(5x)]\,(2x)}{(1+x^2)^2}.$$

(b) $g(x) = \ln\left(e^{x^2} + \sqrt{1+x^4}\right)$

$$g'(x) = \frac{1}{e^{x^2} + \sqrt{1+x^4}}\left[e^{x^2}(2x) + \frac{(4x^3)}{2\sqrt{1+x^4}}\right].$$

[4] **2.** Find an equation for the line that is tangent to this curve at the point where $x = 1$:

$$y = \ln\left(\frac{2x-1}{2x+1}\right).$$

Simplify $y = \ln(2x-1) - \ln(2x+1)$. Thus

$$y' = \frac{2}{2x-1} - \frac{2}{2x+1}, \quad \text{giving} \quad y'(1) = 2 - \frac{2}{3} = \frac{4}{3}.$$

When $x = 1$, $y(1) = \ln(1) - \ln(3) = -\ln(3)$. The tangent line is

$$y = y(1) + y'(1)[x-1], \quad \text{i.e.,} \quad y = -\ln(3) + \frac{4}{3}(x-1).$$

Continued on page 4

[5] **3.** Find an equation for the line tangent to this curve at the point $(2, 1)$:

$$x^2y^3 + x^3 - y^2 = 11.$$

Implicit differentiation gives $\qquad 2xy^3 + x^2(3y^2y') + 3x^2 - 2yy' = 0.$

Let m denote the slope of the tangent line at (2,1). Substitution gives

$$4 + 12m + 12 - 2m = 0, \quad \text{i.e.,} \quad m = -\frac{16}{10} = -\frac{8}{5}.$$

The tangent line has equation $\qquad y = 1 + m(x - 2) = 1 - \frac{8}{5}(x - 2).$

[7] **4.** Let $f(x) = \frac{4}{\pi}\arctan(2x)$, and define $\qquad \alpha = f\left(\frac{1}{2} + \frac{\pi}{10^{10}}\right).$

Clearly, $\alpha \approx f(\frac{1}{2})$ and $f(\frac{1}{2}) = 1$. (Alternative notation for arctan is $\tan^{-1}$.)

(a) Find a more accurate approximation for α.

(b) Decide if your improved approximation is larger or smaller than the exact value of α. Explain.

(a) Recall $\frac{d}{dt}\arctan(t) = \frac{1}{1+t^2}$. By the Chain Rule,

$$f'(x) = \frac{4}{\pi} \cdot \frac{2}{1 + (2x)^2}, \quad \text{so} \quad f'(\tfrac{1}{2}) = \frac{4}{\pi}.$$

The line tangent to the curve $y = f(x)$ at the point where $x = \frac{1}{2}$ is

$$y = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})[x - \tfrac{1}{2}] = 1 + \frac{4}{\pi}[x - \tfrac{1}{2}].$$

Using the tangent to approximate the graph suggests

$$f(\alpha) \approx 1 + \frac{4}{\pi}\left[\left(\frac{1}{2} + \frac{\pi}{10^{10}}\right) - \frac{1}{2}\right] = 1 + 4 \times 10^{-10} = 1.0000000004.$$

(b) Differentiating $f'(x) = (8/\pi)[1 + 4x^2]^{-1}$ gives

$$f''(x) = -\frac{8}{\pi} \cdot \frac{8x}{(1 + 4x^2)^2}.$$

Clearly $f''(x) < 0$ whenever $x > 0$, so the curve $y = f(x)$ is concave down in the interval between $\frac{1}{2}$ and α. Therefore the tangent line lies above the curve in this region, and the approximate value found above is a little larger than α.

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[6] **5.** Find each limit below or show that it does not exist. Justify your results with algebra, not with your calculator!

(a) $\lim_{x \to 0} \left(\dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x} \right)$

One approach is to recognize the definition of the derivative for a certain function:

$$\lim_{x \to 0} \left(\frac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x} \right) = \frac{d}{dx}\left[\frac{1}{2+x}\right]_{x=0} = \left[-\frac{1}{(2+x)^2}\right]_{x=0} = -\frac{1}{4}.$$

A direct approach works, too:

$$\lim_{x \to 0} \frac{1}{x}\left(\frac{1}{2+x} - \frac{1}{2}\right) = \lim_{x \to 0} \frac{1}{x}\left(\frac{2-(2+x)}{2(2+x)}\right) = \lim_{x \to 0}\left(\frac{-x}{2x(2+x)}\right) = -\frac{1}{4}.$$

(b) $\lim_{x \to \infty} \left(\sqrt{x^2+cx} - x\right)$, where c is a constant. (Answer in terms of c.)

Conjugation cracks this. The given limit equals

$$\lim_{x \to \infty} \sqrt{\frac{x^2+cx-x}{1}}\left(\frac{\sqrt{x^2+cx}+x}{\sqrt{x^2+cx}+x}\right) = \lim_{x \to \infty}\left(\frac{[x^2+cx]-x^2}{\sqrt{x^2+cx}+x}\right)$$

$$= \lim_{x \to \infty}\left(\frac{c}{\sqrt{1+c/x}+1}\right) = \frac{c}{2}.$$

Continued on page 6

[6] **6.** Identify the largest number in the sequence

$$a_1 = 1,\ a_2 = 2^{1/2},\ a_3 = 3^{1/3},\ a_4 = 4^{1/4},\ \ldots,\ a_n = n^{1/n},\ \ldots.$$

Hint: Calculating each number in this infinite list would take forever. Analyzing the function $f(x) = x^{1/x}$ on a suitable domain is one way to reduce the work required.

The hint is relevant because $a_n = f(n)$ for the function $f(x) = x^{1/x}$. Take logs to produce the identity $\ln f(x) = x^{-1}\ln(x)$, then differentiate to get

$$\frac{f'(x)}{f(x)} = -x^{-2}\ln(x) + x^{-2} = \frac{1-\ln(x)}{x^2}.$$

In the interval where $x > 1$, we have $f(x) > 0$, so the sign of $f'(x)$ agrees with the sign of $1 - \ln(x)$. Therefore $f'(x) > 0$ for $1 < x < e$ and $f'(x) < 0$ for $e < x < \infty$. Since $2 < e < 3$, we deduce that

$$f(1) < f(2) \quad \text{and} \quad f(3) > f(4) > f(5) > \cdots.$$

The only contenders for the maximum value are $f(2)$ and $f(3)$. The calculator can confirm that $f(3) > f(2)$, so $a_3 = 3^{1/3}$ is the largest number in the given sequence.

Continued on page 7

[8] **7.** A particle moves along the x-axis, where position is measured in metres. At time $t \geq 0$, measured in seconds, the particle's acceleration is

$$a = 2t - 3.$$

At time $t = 0$, the particle has position $x = 10$ and velocity $v = -4$.

(a) At some time $t > 0$, the particle's direction of motion changes. Find the particle's position at this instant.

(b) Find the total distance travelled by the particle during the first 6 seconds of its motion.

Let v denote the particle's velocity. Then

$$\dot{v}(t) = a(t) = 2t - 3 \implies v(t) = t^2 - 3t + v_0 \quad \text{for some constant } v_0;$$
$$-4 = v(0) = v_0 \implies v(t) = t^2 - 3t - 4 = (t-4)(t+1).$$

Likewise, if x denotes the particle's position, we have

$$\dot{x}(t) = v(t) = t^2 - 3t - 4 \implies x(t) = \frac{1}{3}t^3 - \frac{30}{2}t^2 - 4t + x_0 \quad \text{for some } x_0;$$
$$10 = x(0) = x_0 \implies x(t) = \frac{1}{3}t^3 - \frac{30}{2}t^2 - 4t + 10.$$

(a) The particle is moving to the left when $0 < t < 4$ (because $v(t) < 0$) and to the right when $4 < t$ (because $v(t) > 0$). Its direction of motion changes when $t = 4$: its position at that instant is

$$x(4) = -\frac{26}{3} \approx -8.67.$$

(b) "Total distance travelled" accounts for the change at $t = 4$:

$$s = [x(0) - x(4)] + [x(6) - x(4)] = \frac{56}{3} + \frac{38}{3} = \frac{94}{3} \approx 31.33.$$

Continued on page 8

[9] **8.** Newton's Best Coffee (NBC) serves a brew that's too hot to drink immediately. Twenty (20) minutes after a cup is served, its temperature is 70°C; waiting another two (2) minutes lets the temperature drop to 68°C. A visitor suggests that since the temperature has dropped two degrees in two minutes, the coffee must have been 90°C when it was served.

(a) Explain in words, with no equations or calculations, why this reasoning is not perfectly accurate.

(b) Decide whether the true serving temperature was higher or lower than 90°C. Explain your decision in words, with no equations or calculations.

(c) Assuming the room temperature at NBC is 20°C, calculate the coffee's actual serving temperature.

(a) The rate of change of temperature is not constant, even under ideal conditions.

(b) Newton's Law of Cooling says the rate of change of temperature is proportional to the difference between the coffee temperature and the ambient temperature. That difference is larger when the coffee is served than it is 20 minutes later, so the rate of change observed after 20 minutes consistently underestimates the rate at all earlier times. The coffee must have started out hotter than 90°C.

(c) Let $T(t)$ denote the coffee temperature and T_s the temperature of the surrounding environment. Let $u(t) = T(t) - T_s$. By Newton's Law, $du/dt = -ku$ for some constant k, so

$$u(t) = Ae^{-kt}, \quad \text{or} \quad T(t) = T_s + Ae^{-kt}.$$

Given $T_s = 20$, we have

$$\left.\begin{aligned} 70 &= T(20) = 20 + Ae^{-20k} \\ 68 &= T(22) = 20 + Ae^{-22k} \end{aligned}\right\} \implies \frac{70-20}{68-20} = \frac{Ae^{-20k}}{Ae^{-22k}} = e^{2k}.$$

Hence $k = \frac{1}{2}\ln\left(\frac{25}{24}\right) \approx 0.02041$, and $A = 50e^{20k} = 50\left(\frac{25}{24}\right)^{10}$.

The serving temperature (in degrees C) was

$$T(0) = 20 + A = 20 + 50\left(\frac{25}{24}\right)^{10} \approx 95.2.$$

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[6] **9.** Consider the function $f(x) = x\sin(|x|)$.

(a) Does $f'(0)$ exist? If so, explain why and calculate it; if not, explain why not.

(b) Does $f''(0)$ exist? If so, explain why and calculate it; if not, explain why not.

(a) By definition,

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin|h|}{h} = \lim_{h\to 0}\sin|h| = 0.$$

Therefore $f'(0)$ exists and its value is 0.

(b) To make a similar calculation of $f''(0)$, we need to know $f'(h)$ when h is small in magnitude.

In the interval $(0,\infty)$, $f(x) = x\sin(x)$, so the product rule gives

$$f'(x) = \sin(x) + x\cos(x) \qquad \text{when } x > 0.$$

In the interval $(-\infty, 0)$, $f(x) = x\sin(-x) = -x\sin(x)$, so the calculation above can be re-used:

$$f'(x) = -\sin(x) - x\cos(x) \qquad \text{when } x < 0.$$

Recalling $f'(0) = 0$ from part (a), we investigate

$$f''(0) = \lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h} = \lim_{h\to 0}\frac{f'(h)}{h},$$

using two one-sided limits. From the right,

$$\lim_{h\to 0^+}\frac{\sin(h)+h\cos(h)}{h} = \lim_{h\to 0^+}\left[\frac{\sin(h)}{h}+\cos(h)\right] = 2.$$

From the left, by similar methods,

$$\lim_{h\to 0^-}\frac{-\sin(h)-h\cos(h)}{h} = \lim_{h\to 0^-}\left[-\frac{\sin(h)}{h}-\cos(h)\right] = -2.$$

Since these results disagree, $f''(0)$ does not exist.

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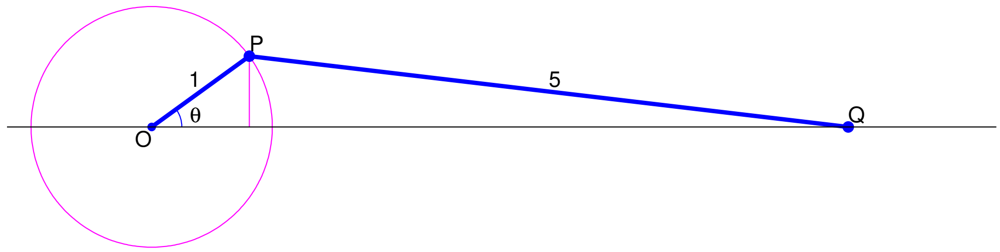
[9] **10.** A wheel of radius 1 metre spins counterclockwise around the origin at a constant speed of 10 revolutions per second. One end of a rod 5 metres long pivots on a point P on the wheel's perimeter; the rod's other end, Q, slides back and forth along the x-axis. See the sketch. Find the linear speed of point Q at the instant when the angle θ shown in the sketch is $\pi/4$ radians.



(*Hint*: Split $|\overline{OQ}| = u + w$, where u and w are formed by dropping a perpendicular from P to $\overline{OQ}$. You can find u and w from basic trigonometry.)

The legs of the right triangle in the unit circle have lengths $u = \cos\theta$, $y = \sin\theta$. (Please label the sketch above.) So for u and w as in the hint, Pythagoras gives $w = \sqrt{5^2 - y^2} = \sqrt{5^2 - \sin^2\theta}$ and

$$|\overline{OQ}| = \cos\theta + \sqrt{5^2 - \sin^2\theta}.$$

Therefore, taking the time derivative of both sides,

$$\frac{d}{dt}|\overline{OQ}| = -\sin\theta\left(\frac{d\theta}{dt}\right) + \frac{-2\sin\theta\cos\theta}{2\sqrt{5^2 - \sin^2\theta}}\left(\frac{d\theta}{dt}\right).$$

When $\theta = \pi/4$, we have

$$\frac{d}{dt}|\overline{OQ}| = \left(-\frac{\sqrt{2}}{2} - \frac{1/2}{\sqrt{49/2}}\right)\left(\frac{d\theta}{dt}\right) = -\frac{4\sqrt{2}}{7}\left(\frac{d\theta}{dt}\right).$$

There are 2π radians in one revolution, so $\dfrac{d\theta}{dt} = 10(2\pi) = 20\pi$ radians per second. We deduce

$$\frac{d}{dt}|\overline{OQ}| = -\frac{4\sqrt{2}}{7}(20\pi) \approx -50.8.$$

The linear speed of Q (relative to O) is about 50.8 m/s. ("Speed" can't be negative.)

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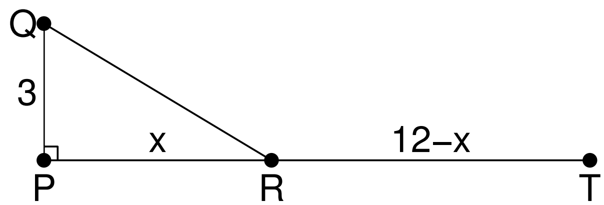
[9] **11.** Residents of island Q need a new fibre-optic cable for their network. The island is 3 km offshore from the nearest point P on a straight coastline, and the nearest broadband signal source is in town T, 12 km along the shore from P. See the sketch. Underwater cable costs twice as much as dry-land cable, so the islanders decide to save money by running underwater cable from Q to R and dry-land cable from R to T. What location for point R gives the lowest cost?



Taking the definition of x suggested in the sketch, Pythagoras gives the length of underwater cable: $\sqrt{9+x^2}$. The total cost of cable along the path QRT is proportional to

$$f(x) = 2\sqrt{9+x^2} + (12-x).$$

We seek to minimize $f(x)$ over the interval $0 \le x \le 12$. Here

$$f'(x) = \frac{2x}{\sqrt{9+x^2}} - 1 = \frac{2x - \sqrt{9+x^2}}{\sqrt{9+x^2}}.$$

In the region where $x > 0$, we have

$$f'(x) = 0 \iff 2x = \sqrt{9+x^2} \iff 4x^2 = 9 + x^2 \iff x^2 = 3$$
$$\iff x = \sqrt{3}.$$

The simplified form of f' above reveals that $f'(x) < 0$ for $0 < x < \sqrt{3}$ and $f'(x) > 0$ for $x > \sqrt{3}$, so the point $x = \sqrt{3}$ gives an absolute minimum value for f in the domain of interest.

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[9] **12.** A particle moves along a vertical line, starting at time $t = 0$ and finishing at time $t = 3$. Its height at time t is

$$y = 4t^3 - 24t^2 + 21t, \qquad 0 \le t \le 3.$$

Find the highest and lowest points reached by this particle, and find when its speed is greatest. Give full reasons for your conclusions.

Calculation gives

$$v = \frac{dy}{dt} = 12t^2 - 48t + 21 = 3(4t^2 - 16t + 7) = 3(2t - 7)(2t - 1),$$
$$a = \frac{dv}{dt} = 24t - 48 = 24(t - 2).$$

The highest and lowest points will occur either at a critical point of y or at an endpoint of its domain. A critical point for y is a root of $y' = v$: the only one in the interval $[0, 3]$ is at $t = \frac{1}{2}$. The contenders and results are

$$\begin{aligned} y(0) &= 0 \\ y(\tfrac{1}{2}) &= 5 && \ldots \text{ the highest point,} \\ y(3) &= -45 && \ldots \text{ the lowest point.} \end{aligned}$$

The extreme velocities will occur either at a critical point of v or at an endpoint of its domain. A critical point for v is a root of $v' = a$: the only one in the interval $[0, 3]$ is at $t = 2$. So the velocities of interest are

$$v(0) = 21, \qquad v(2) = -27, \qquad v(3) = -15.$$

Now *speed is the magnitude of velocity,* so the particle's maximum speed in the given interval is $|v(2)| = 27$.

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[9] **13.** Using the axes provided on the next page, make a reasonable sketch of the curve

$$y = 4x + 2 - 5\ln(1 + x^2). \qquad (\textit{Hint}:\ (1 + x^2)^2 y'' = 10(x^2 - 1).)$$

Support your sketch with calculations that identify the following features:

(a) The exact (x, y) coordinates of each critical point.

(b) Exact intervals on which the curve is increasing or decreasing.

(c) The exact (x, y) coordinates of each inflection point.

(d) Is it correct to say, "The line $y = 4x + 2$ is a slant asymptote for this curve?" Why or why not?

(a) Here $$y' = 4 - 5\left(\frac{2x}{1 + x^2}\right) = \frac{4 - 4x^2 - 10x}{1 + x^2}.$$

Factoring reveals that $x = \frac{1}{2}$ and $x = 2$ are critical points:

$$y' = \frac{-2(2x - 1)(x - 2)}{1 + x^2}.$$

The corresponding points on the curve are

$$\left(\frac{1}{2}, 4 - 5\ln(5/4)\right), \qquad (2, 10 - 5\ln(5)).$$

(b) One has $y' < 0$ when $\frac{1}{2} < x < 2$, so the curve is decreasing on this interval. On the interval $(-\infty, \frac{1}{2})$ it is increasing; likewise for the interval $(2, +\infty)$.

(c) The hint saves direct calculation: $$y'' = \frac{10(x - 1)(x + 1)}{(1 + x^2)^2}.$$

This has two zeros, at $x = -1$ and $x = 1$. Between them one has $y'' < 0$, so the curve is concave down. In the two complementary intervals, $y'' > 0$, so the curve is concave up. Therefore we have true inflection points in both locations, with exact coordinates

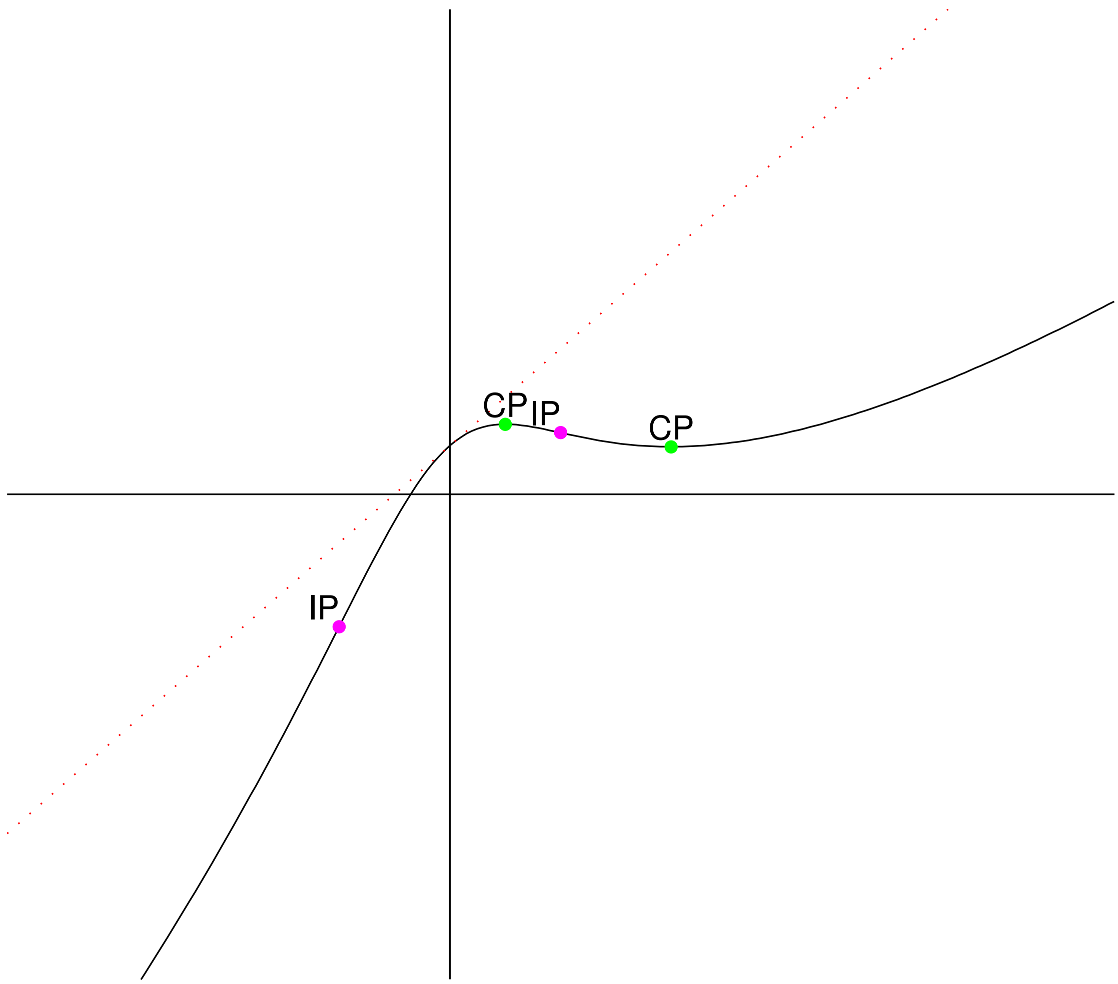
$$(-1, -2 - 5\ln(2)), \qquad (1, 6 + 5\ln(2)).$$

(d) The line $y = 4x + 2$ is not a slant asymptote for the given curve because the difference between the y-values on the curve and the y-values on the line is $5\ln(1 + x^2)$, and this does not have limit 0 as either $x \to -\infty$ or $x \to +\infty$.

Continued on page 14

Here is a computerized sketch of the curve. The labelling here is inadequate: students are expected to do better.

The line $y = 4x + 2$ is tangent to the curve at the point $(0, 2)$, and since $5\ln(1 + x^2) \geq 0$ for all real x, the curve always lies below it. This helps guide the production of the sketch below. (The line $y = 4x + 2$ is shown here for reference only: it's *not an asymptote,* as noted in part (d).)



Continued on page 15

[7] **14.** Eddie is in a hurry to find the area A that lies above the x-axis and below the curve

$$y = \frac{(a^2 - x^2)}{a}.$$

(Here $a > 0$ is a constant.) Observing that the curve is concave down and passes through three points that also lie on the semicircle $y = \sqrt{a^2 - x^2}$, Eddie decides to approximate A using the area between the x-axis and the semicircle.

(a) Is Eddie's approximation larger or smaller than the true value of A?

(b) Find the exact value of A and use it to calculate the percentage error in Eddie's approximation.

(a) The circle has vertical tangent lines when it meets the x-axis, whereas the parabola has oblique ones. Thus it's reasonable to guess that the semicircle's area will be larger than A. Of course, we'll soon find out.

(b) The exact area is

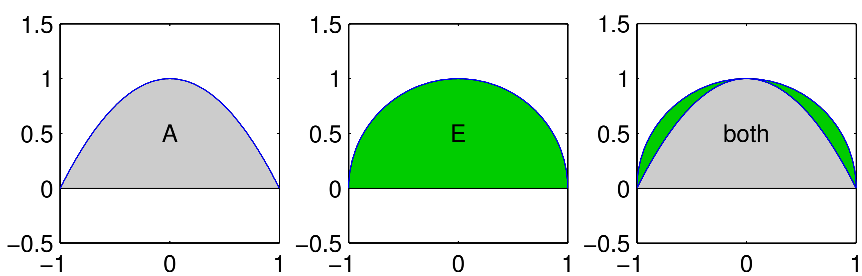
$$A = \int_{-a}^{a} \left(\frac{a^2 - x^2}{a} \right) dx = \frac{1}{a} \left[\frac{a^2 x - x^3}{3} \right]_{x=-a}^{a} = \frac{4}{3} a^2.$$

Eddie's approximation is $E = \frac{1}{2}\pi a^2$. The relative error in this approximation is

$$\frac{E - A}{A} = \frac{\frac{\pi}{2}a^2 - \frac{4}{3}a^2}{\frac{4}{3}a^2} = \frac{3\pi}{8} - 1 \approx 0.1781.$$

Thus E is too big by about 18%.

Here are some sketches showing the situation when $a = 1$. (Sketches are not required for credit.)



The End