A. The Sturm-Liouville Family of Problems

Given a real interval \([a, b]\), functions \(p(x), q(x), r(x)\), and constants \(c_0, c_1, d_0, d_1\), consider

\[
\begin{align*}
\text{(ODE)} & \quad (p(x)y'(x))' - q(x)y(x) + \lambda r(x)y(x) = 0, \quad a < x < b, \\
\text{BC}(a) & \quad c_0y(a) + c_1y'(a) = 0, \\
\text{BC}(b) & \quad d_0y(b) + d_1y'(b) = 0.
\end{align*}
\]

(EVP)

Assume always that \(p(x) > 0\) and \(r(x) > 0\) for all \(x \in [a, b]\), and that \((c_0, c_1) \neq (0, 0), (d_0, d_1) \neq (0, 0)\). This is the prototype for a Sturm-Liouville eigenvalue problem.

**Terminology.**

(a) A constant \(\lambda\) is an eigenvalue in (EVP) if using this particular value in (ODE) allows for some nonzero solution function \(y\) in (ODE)+(BC). (Any such function is called an eigenfunction corresponding to \(\lambda\).)

(b) A nonzero function \(y\) defined on \([a, b]\) is an eigenfunction in (EVP) if this particular function satisfies both (BC) and (ODE) for some choice of constant \(\lambda\). (That constant is then called the eigenvalue corresponding to \(y\).)

**Practicalities.**

(a) Testing some given number \(\lambda\) for the status of “eigenvalue in (EVP)” is not easy. Typically it requires solving the differential equation (ODE) and applying the boundary conditions (BC) to see if a nontrivial function is compatible. We have seen examples, and there will be more.

(b) Testing some given function \(y\) for the status of “eigenfunction in (EVP)” is very easy. Checking the boundary values and verifying (BC) is extremely simple, and substitution into (ODE) reduces that identity to an equation in which the constant \(\lambda\) is the only unknown. Alternatively, one could rearrange (ODE) like this:

\[
\lambda = \frac{q(x)y(x) - (p(x)y'(x))'}{r(x)y(x)}, \quad a < x < b.
\]

(*)

When \(y(x)\) is given, everything on the right side is known. If that ratio simplifies to produce a constant, then \(y\) is an eigenfunction and the ratio is the eigenvalue; if not, then \(y\) is not an eigenfunction.

As a consequence of (a)–(b) here, it is often enough to remember only the eigenfunctions in (EVP). The eigenvalues are always easily accessible from a quick calculation analogous to (*).

**Example.** When the interval has \(a = 0\) and the coefficient functions are the constants \(p(x) = 1, q(x) = 0, r(x) = 1\), the ODE becomes to get \(y'' + \lambda y = 0\). Making the further choices \((c_0, c_1) = (1, 0)\) and \((d_0, d_1) = (1, 0)\) turns the BC’s into \(y(0) = 0 = y(b)\). Thus the eigenvalue problem that produces the Fourier Sine Series fits into the
framework above. Each function $y_n(x) = \sin(n\pi x/b)$ is an eigenfunction, and the corresponding eigenvalue comes from

$$\lambda_n = -\frac{y_n''(x)}{y(x)} = -\frac{(n\pi/b)^2 \sin(n\pi x/b)}{\sin(n\pi x/b)} = \frac{n^2\pi^2}{b^2}.$$  

The eigenvalue problems associated with the FCS, HPSS, and HPCS have Sturm-Liouville form as well—they just involve different choices of $(c_0, c_1)$ and $(d_0, d_1)$.

**Derivative Packing.** In (EVP), all the derivatives are concentrated in the first term, $(py')'$. Using the product rule leads to this equivalent equation:

$$p(x)y''(x) + p'(x)y'(x) - q(x)y(x) + \lambda r(x)y(x) = 0, \quad a < x < b.$$  

The special relationship between the coefficients of $y''$ and $y'$ here is critical in deriving the following list of facts. (See below for details.) Notice how these correspond to known properties of the eigenfunction families we have already worked with.

**Theorem (Sturm-Liouville).** Under the stated conditions governing (EVP),

(a) All the eigenvalues are real numbers.
(b) The eigenvalues form an infinite sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ such that

$$\lim_{n \to \infty} \lambda_n = +\infty.$$  

c) For each eigenvalue $\lambda_n$, all solutions of (ODE)/(BC) with $\lambda = \lambda_n$ can be expressed as constant multiples of a single representative eigenfunction $y_n$. [That is, the eigenspace $E(\lambda_n)$ is one-dimensional.]

d) Eigenfunctions with distinct eigenvalues are “orthogonal with weight $r$”. That is, if $y_m, y_n$ are eigenfunctions for eigenvalues $\lambda_m, \lambda_n$,

$$(y_m, y_n)_r \overset{\text{def}}{=} \int_a^b y_m(x)y_n(x)r(x) \, dx = 0 \quad \text{whenever } \lambda_m \neq \lambda_n.$$  

Note the presence of $r(x)$ in this orthogonality relation.

e) Eigenfunction series give arbitrarily good mean-square approximations for any Riemann-integrable $f: [a, b] \to \mathbb{R}$.

**Sturm-Liouville/Self-Adjoint Form.** When all derivatives are condensed into the term $(py')'$ as shown in (ODE), the differential equation is said to be in “Sturm-Liouville Form” or “Self-Adjoint Form”. This arrangement is ideal for integration by parts. Here’s why. Suppose $u(x)$ and $v(x)$ are any differentiable functions defined on $[a, b]$. Integration by parts gives

$$\int_a^b u(x) \left(p(x)v'(x)\right)' \, dx = u(x)(p(x)v'(x)) \bigg|_a^b \quad - \int_a^b (p(x)v'(x))u'(x) \, dx. \quad (1)$$  

Swapping the letters $u$ and $v$ leads to a different-looking, but equivalent, statement:

$$\int_a^b v(x) \left(p(x)u'(x)\right)' \, dx = v(x)(p(x)u'(x)) \bigg|_a^b \quad - \int_a^b (p(x)u'(x))v'(x) \, dx. \quad (2)$$  

The integrals on the far right in (1) and (2) are equal, so these two equations can be combined to produce an identity that shows the discrepancy introduced by moving the packaged-derivative operation from one factor to the other:

\[
\int_a^b u(x) \left( (p(x)v'(x))' - (p(x)u'(x))' + v(x) dx \right) = \int_a^b \left( \lambda \right) = \int_a^b \left[ p(x) \left( u(x)v'(x) - u'(x)v(x) \right) \right]_{x=a}^b. \tag{3}
\]

If the functions \( u \) and \( v \) happen to satisfy both BC\((a)\) and BC\((b)\), the bracketed terms in (3) equal 0 [exercise!] and the identity becomes

\[
\int_a^b u(x)(p(x)v'(x))' dx = \int_a^b (p(x)u'(x))'v(x) dx \quad \text{(assuming (BC))}.
\]

In particular, if we choose eigenfunctions \( y_m \) and \( y_n \) (with eigenvalues \( \lambda_m \) and \( \lambda_n \)) for the functions \( u \) and \( v \) in the calculation above, we can use (ODE) to replace the packaged derivative terms:

\[
\int_a^b y_m(x) \left( p(x)y_n'(x) \right)' dx = \int_a^b \left( p(x)y_m'(x) \right)' y_n(x) dx
\]

\[
\int_a^b y_m(x) \left( q(x)y_n(x) - \lambda_n r(x)y_n(x) \right) dx = \int_a^b \left( q(x)y_m(x) - \lambda_m r(x)y_m(x) \right) y_n(x) dx
\]

\[
-\lambda_n \int_a^b y_m(x)y_n(x)r(x) dx = -\lambda_m \int_a^b y_m(x)y_n(x)r(x) dx.
\]

This is equivalent to

\[
(\lambda_m - \lambda_n) \int_a^b y_m(x)y_n(x)r(x) dx = 0. \tag{4}
\]

If we have \( \lambda_m \neq \lambda_n \), this forces the integral to equal 0, and that proves statement \((d)\) in the Sturm-Liouville theorem.

**Exercise Solution.** To handle the evaluated terms shown in line (3), rewrite the statement that both functions \( u \) and \( v \) satisfy BC\((a)\) as the pair of equations

\[
c_0 u(a) + c_1 u'(a) = 0
\]

\[
c_0 v(a) + c_1 v'(a) = 0.
\]

In matrix form, this is equivalent to

\[
\begin{bmatrix}
  u(a) & u'(a) \\
  v(a) & v'(a)
\end{bmatrix}
\begin{bmatrix}
c_0 \\
  c_1
\end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.
\]

Since \((c_0, c_1) \neq (0, 0)\), this equation can only hold if the matrix on the left is non-invertible. That is, we must have

\[
0 = \det \left( \begin{bmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{bmatrix} \right) = u(a)v'(a) - u'(a)v(a).
\]

This shows that the evaluated term in (3) vanishes at the left endpoint, \( x = a \). The same line of argument applies at the right endpoint, \( x = b \).
Calculation (1) can be interesting even when \( u \) and \( v \) are equal. Suppose \( y_n \) is some eigenfunction for (EVP). Using \( u = y_n = v \), and exploiting the identity in (ODE), we have

\[
\int_a^b y_n(x) \left( p(x)y'_n(x) \right)' \, dx = p(x)y_n(x)y'_n(x) \bigg|_{x=a}^{x=b} - \int_a^b p(x)y''_n(x) \, dx
\]

\[
\int_a^b y_n(x) \left( q(x)y_n(x) - \lambda r(x)y_n(x) \right) \, dx = p(x)y_n(x)y'_n(x) \bigg|_{x=a}^{x=b} - \int_a^b p(x)y''_n(x) \, dx
\]

Rearranging this produces an equation involving the eigenvalue \( \lambda_n \):

\[
\lambda_n = \frac{\int_a^b \left( p(x)y'_n(x)^2 + q(x)y_n(x)^2 \right) \, dx - \left[ p(b)y_n(b)y'_n(b) - p(a)y_n(a)y'_n(a) \right]}{\int_a^b r(x)y_n(x)^2 \, dx}. \tag{5}
\]

To see how useful this can be, consider the simple constant functions \( p(x) = 1 \), \( q(x) = 0 \), \( r(x) = 1 \) and boundary conditions \( y(a) = 0 \), \( y(b) = 0 \) that we associate with Fourier Sine Series. Without knowing anything about the eigenfunction \( y_n \), we can see that its corresponding eigenvalue is

\[
\lambda_n = \frac{\int_a^b y'_n(x)^2 \, dx}{\int_a^b y_n(x)^2 \, dx}. \tag{6}
\]

The integrals on the top and bottom of this fraction must both be non-negative, so \( \lambda_n \geq 0 \) is guaranteed. In fact, \( \lambda_n = 0 \) would require \( y'_n(x) = 0 \) for all \( x \), and that (together with the BC’s) would force \( y_n \) be the constant function 0—which is incompatible with the hypothesis that \( y_n \) is nontrivial in the first place. So we can’t have \( \lambda_n = 0 \) after all. Therefore \( \lambda_n > 0 \) must hold for each \( n \). Of course this is no surprise for the FSS example, but the possibility of predicting eigenvalue signs in problems were the form of the eigenfunctions is not known in advance can be a real asset.

**Practice.** In the Trench textbook, try exercise 13.2 #8.

**Arranging Sturm-Liouville Form.** Differential equations encountered “in the wild” may not have all their derivatives packed into the first term as shown in (ODE). To reorganize a given equation into this form, use the expanded version (0) as a target:

\[
p(x)y'' + p'(x)y' - q(x)y + \lambda r(x)y = 0.
\]

That is, given any second-order linear ODE, try to match it with the form above by introducing a new unknown \( p(x) \) and then matching the coefficients of \( y'' \) and \( y' \).

**Example.** Find orthogonality weight for \( y'' + 4y' + (4 + 9\lambda)y = 0 \).
Solution. Multiply the given equation by \( p(x) \). That makes the \( y'' \) coefficient exactly what we want:
\[
p(x)y'' + 4p(x)y' + (4 + 9\lambda)p(x)y = 0.
\]
To match the coefficient on \( y' \), we want \( p'(x) = 4p(x) \). This is a simple differential equation for \( p(x) \). The general solution is \( p(x) = Ae^{4x}, A \in \mathbb{R} \). Any constant \( A \) will work; we take \( A = 1 \) because it’s simple. (Admittedly, \( A = 0 \) would be simpler, but that choice would reduce our original differential equation to “0 = 0” — a profound loss of information about \( y \)!) Then the given equation is equivalent to
\[
0 = e^{4x}y'' + 4e^{4x}y' + (4 + 9\lambda)e^{4x}y = (e^{4x}y')' + 4e^{4x}y + \lambda(9e^{4x})y = 0.
\]
It follows that eigenfunctions on the interval \( a < x < b \), with distinct eigenvalues, are orthogonal with weight \( r(x) = 9e^{4x} \):
\[
\int_a^b y_m(x)y_n(x)9e^{4x}dx = 0 \quad \text{whenever } m \neq n.
\]
(You can drop the factor 9 here and the statement remains true.) // /

Example. Find the orthogonality weight for Laguerre’s equation,
\[
xy'' + (1-x)y' + \lambda y = 0.
\]
Solution. Know the terminology: In the notation of (EVP) above, “orthogonality weight” means “function \( r(x) \)”. Multiply through Laguerr’s equation by \( \frac{p(x)}{x} \) to match the \( y'' \) term with the form in (0):
\[
p(x)y'' + \left(\frac{1-x}{x}\right)p(x)y' + \lambda\left(\frac{p(x)}{x}\right)y = 0.
\]
To match the \( y' \)-term, we need \( p'(x) = \frac{1-x}{x}p(x) \), i.e.,
\[
\frac{dp}{p} = \frac{1-x}{x}dx \iff \ln(p) = \ln x - x + C \iff p(x) = Axe^{-x}.
\]
Choose \( A = 1 \) for simplicity, to get
\[
0 = xe^{-x}y'' + (1-x)e^{-x}y' + \lambda e^{-x}y = (xe^{-x}y')' + \lambda e^{-x}y.
\]
Hence the orthogonality weight is \( r(x) = e^{-x} \). This function is not clearly visible in the given equation, but it’s very important. In any version of (EVP) based on this ODE with some interval \([a, b]\), eigenfunctions \( y_m \) and \( y_n \) will have the orthogonality property
\[
\int_a^b y_m(x)y_n(x)e^{-x}dx = 0 \quad \text{whenever } m \neq n.
\]
(Think about this: the statement just made only works if 0 is not a point in the interval \([a, b]\). Why?) // //
**Practice.** In the Trench textbook, try exercises 13.2 #1–7.

**Example (Euler Series from Laplace Eqn).** For the following eigenvalue problem based on an Euler-type equation,

\[
x^2y''(x) + xy'(x) + \lambda y(x) = 0, \quad 1 < x < b, \\
y(1) = 0 = y(b),
\]

the eigenfunctions turn out to be (practice!) \( y_n(x) = \sin\left(\frac{n\pi}{\ln b} \ln x\right) \), with corresponding eigenvalues

\[
\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2, \quad n = 1, 2, 3, \ldots.
\]

From FSS theory plus change of variables, or from general SL theory, we have the orthogonality relation

\[
0 = \int_1^b y_m(x)y_n(x) \frac{dx}{x}, \quad m \neq n.
\]

**Orthogonality and Coefficient Extraction.** Suppose we know a full list of representative eigenfunctions \( y_1, y_2, \ldots \) in problem (EVP). Then, let \( f(x) \) be given on \([a, b]\). What coefficients \( c_n \) will make a correct eigenfunction series identity below?

\[
f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad a < x < b. \tag{*}
\]

Orthogonality provides the answer. Pick any particular subscript \( k \), multiply (*) by \( y_k(x)r(x) \) and integrate:

\[
\int_a^b f(x)y_k(x)r(x) \, dx = \sum_{n=1}^{\infty} c_n \int_a^b y_n(x)y_k(x)r(x) \, dx = c_k \int_a^b y_k(x)^2r(x) \, dx.
\]

Thanks to orthogonality, all terms in the sum except the one where \( n = k \) equal zero. This explains the second step above, and leads to

\[
c_k = \frac{\int_a^b f(x)y_k(x)r(x) \, dx}{\int_a^b y_k(x)^2r(x) \, dx}.
\]

**Normalization.** When choosing the eigenfunction \( y_k \) for \( \lambda_k \), if you set the constants right you can arrange

\[
\int_a^b y_k(x)^2r(x) \, dx = 1. \tag{**}
\]

This makes the formula for \( c_k \) above a little cleaner. Eigenfunctions chosen to satisfy (**) are called **normalized**. In our course, **normalizing eigenfunctions is a waste of time**: it makes no difference to the series solutions we calculate. Don’t bother!
Consider this eigenvalue problem:

\[(\text{ODE}) \quad x^2y'' + xy' + \lambda y = 0, \quad 1 < x < e,\]
\[(\text{BC}) \quad y(1) = 0 = y(e).\]

(a) Express the equation in standard form and find the orthogonality relation.

Get \(r(x) = 1/x\), deduce that if \(y_m\) and \(y_n\) are eigenfunctions with different eigenvalues, then

\[0 = \int_{1}^{e} y_m(x)y_n(x) \frac{dx}{x}.\]

(b) Find all eigenvalues and eigenfunctions.

Grinding case-by-case analysis leads to

\[\lambda_n = n^2\pi^2, \quad y_n(x) = \sin(n\pi \ln x), \quad n = 1, 2, \ldots.\]

(c) [Task 1: Change of Basis/Coefficient Extraction] Given \(f(x)\), find formulas for the coefficients in

\[f(x) = \sum_{n=1}^{\infty} c_n y_n(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi \ln x), \quad 1 < x < e.\]

Evaluate the \(c_n\)'s explicitly for \(f(x) = \begin{cases} \ln x, & 1 < x < \sqrt{e}, \\ 0, & \text{otherwise}. \end{cases}\)

Generally,

\[c_n = \frac{(f, y_n)_{x}}{(y_n, y_n)_{x}} = \frac{\int_{1}^{e} f(x) \sin(n\pi \ln x) \cdot x^{-1} \, dx}{\int_{1}^{e} \sin^2(n\pi \ln x) \cdot x^{-1} \, dx}
= 2 \int_{1}^{e} f(x) \sin(n\pi \ln x) \cdot x^{-1} \, dx.\]

For the specific \(f\) in question, splitting gives

\[c_n = 2 \int_{1}^{\sqrt{e}} (\ln x) \sin(n\pi \ln x) \, dx = \int_{0}^{1/2} u \sin(n\pi u) \, du
= \frac{2}{(n\pi)^2} \sin \left( \frac{n\pi}{2} \right) - \frac{1}{n\pi} \cos \left( \frac{n\pi}{2} \right).\]

\[c_n = \begin{pmatrix} 2/\pi^2 & 1/2\pi & -2/9\pi^2 & -1/4\pi & 2/25\pi^2 & 1/6\pi & -2/49\pi^2 & \ldots \end{pmatrix},\]

(d) [Task 2: Equation-Solving] Use a suitable eigenfunction series to solve

\[x^2y'' + xy' + 7y = f(x), \quad 1 < x < e; \quad y(1) = 0 = y(e),\]
with \( f \) as given in part (c) above.

Notice that \( y_n(x) = \sin(n\pi \ln x) \) gives

\[
y_n'(x) = \frac{n\pi}{x} \cos(n\pi \ln x),
\]

\[
y_n''(x) = -\frac{n\pi}{x^2} \cos(n\pi \ln x) - \frac{n^2\pi^2}{x^2} \sin(n\pi \ln x),
\]

so \( x^2 y_n'' + x y_n' = -n^2 \pi^2 \sin(n\pi \ln x) = -n^2 \pi^2 y_n. \)

This recapitulates the eigenvalue/eigenfunction relationship and is useful when guessing \( y = \sum_{n=1}^{\infty} a_n y_n \) for the desired solution. Substitution gives

\[
x^2 y'' + xy' + 7y = \sum_{n=1}^{\infty} a_n \left[ x^2 y_n'' + x y_n' + 7y_n \right]
\]

\[
= \sum_{n=1}^{\infty} a_n \left[ -n^2 \pi^2 + 7 \right] y_n.
\]

Therefore \( a_n \left[ 7 - n^2 \pi^2 \right] \) must match the coefficient \( c_n \) computed above. Answer

\[
a_n = \frac{c_n}{7 - n^2 \pi^2} = \frac{1}{7 - n^2 \pi^2} \left[ \frac{2}{(n\pi)^2} \sin \left( \frac{n\pi}{2} \right) - \frac{1}{n\pi} \cos \left( \frac{n\pi}{2} \right) \right].
\]

(e) Task 3: Dynamics] Solve for \( u = u(x, t) \) in this heat conduction problem:

(PDE) \( u_t = x^2 u_{xx} + xu_x, \quad 1 < x < e, \ t > 0, \)

(BC) \( u(1, t) = 0 = u(e, t), \quad t > 0, \)

(IC) \( u(x, 0) = f(x), \quad 1 < x < e, \)

with \( f \) as above.

Answer: Separation of variables leads to the eigenvalue problem stated above, hence to the postulate

\[
u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi \ln x).
\]

To complete the solution, it suffices to find the functions \( T_n(t) \). First initialize \( T_n(0) = c_n \) (see part (c) above), then propagate \( T_n' = -n^2 \pi^2 T_n \), so

\[
u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 t} \sin(n\pi \ln x),
\]

with \( c_n \) as before.
C. Extended Example 2

Consider this eigenvalue problem:

\begin{align*}
(\text{ODE}) & \quad y'' + \lambda y = 0, \quad 0 < x < 1, \\
(\text{BC}) & \quad y'(0) = 0 = y(1) - y'(1).
\end{align*}

(a) Express the equation in self-adjoint form and find the orthogonality relation.

The equation already is in self-adjoint (Sturm-Liouville) form, with \( p(x) = 1 \), \( q(x) = 0 \), and \( r(x) = 1 \). In particular, any two eigenfunctions \( y_m \) and \( y_n \) having different eigenvalues \( \lambda_m \) and \( \lambda_n \) will obey

\[ \int_0^1 y_m(x)y_n(x) \, dx = 0. \]

(b) Find all eigenvalues and eigenfunctions.

Guessing \( y = e^{sx} \) for constant \( s \in \mathbb{C} \) leads to a solution whenever \( s^2 + \lambda = 0 \). Three types of \( s \)-values emerge, depending on the sign of \( \lambda \).

**Case \( \lambda < 0 \):** Define \( \alpha = \sqrt{-\lambda} > 0 \) to get \( s^2 = \alpha^2 \), so \( s = \pm \alpha \):

\[ y = Ae^{\alpha x} + Be^{-\alpha x}, \quad A, B \in \mathbb{R}. \]

The left BC gives \( 0 = y'(0) = (A - B)\alpha \). Since \( \alpha > 0 \), \( B = A \) and

\[ y = A \left[ e^{\alpha x} + e^{-\alpha x} \right], \quad y' = \alpha A \left[ e^{\alpha x} - e^{-\alpha x} \right]. \quad (\ast) \]

The right BC requires

\[ 0 = y(1) - y'(1) = A \left[ e^{\alpha} + e^{-\alpha} - \alpha(e^{\alpha} - e^{-\alpha}) \right]. \]

For nontriviality (see \( \ast \)) we must have \( A \neq 0 \), so this requires

\[ e^{\alpha} + e^{-\alpha} = \alpha(e^{\alpha} - e^{-\alpha}), \quad \text{i.e.,} \quad \frac{1}{\alpha} = \frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}}. \quad (\ast\ast) \]

Express this last condition as \( L(\alpha) = R(\alpha) \) for the functions \( L \) and \( R \) defined by

\[ L(\alpha) = 1/\alpha, \quad R(\alpha) = \frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}}. \]

Graphical solution for \( \alpha_0 \approx 1.1997 \)
Plotting the curves $z = L(\alpha)$ and $z = R(\alpha)$ for $\alpha > 0$ and looking for an intersection point gives one solution: call this $\alpha_0 \approx 1.1997$, with corresponding eigenvalue $\lambda_0 = -\alpha_0^2 \approx -1.4392$. A representative eigenfunction is obtained by taking $A = 1$ and $\alpha = \alpha_0$ in (*):

$$y_0(x) = e^{\alpha_0 x} + e^{-\alpha_0 x}, \quad \alpha_0 \text{ given by (**)}. $$

**Case $\lambda = 0$:** When $\lambda = 0$ the general solution of the given ODE is $y(x) = A + Bx$. The left BC gives $0 = y'(0) = B$, so $y = A$. The right BC gives $0 = y(1) - y'(1) = A$, so $y = 0$. Only the trivial solution obeys both BC’s, so $\lambda = 0$ is not an eigenvalue.

**Case $\lambda > 0$:** Write $\omega = \sqrt{\lambda} > 0$, so that the characteristic equation is $s^2 = -\omega^2$, with solutions $s = \pm i\omega$. Here

$$y = A \cos(\omega x) + B \sin(\omega x), \quad A, B \in \mathbb{R},$$

and the left BC requires $0 = y'(0) = B\omega$. Since $\omega > 0$, this gives $B = 0$ and

$$y = A \cos(\omega x), \quad y' = -\omega A \sin(\omega x), \quad A \in \mathbb{R}. \quad (\dagger)$$

Now the right BC requires

$$0 = y(1) - y'(1) = A \left[ \cos \omega + \omega \sin \omega \right].$$

For nontriviality (see (\dagger)) we must have $A \neq 0$, so this requires

$$ \cos \omega = -\omega \sin \omega \quad \text{i.e.,} \quad -\frac{1}{\omega} = \frac{\sin \omega}{\cos \omega}. \quad (\ddagger)$$

Express this last condition as $L(\omega) = R(\omega)$ for the functions $L$ and $R$ defined by

$$L(\omega) = -1/\omega, \quad R(\omega) = \tan \omega.$$
Plotting the curves $z = L(\omega)$ and $z = R(\omega)$ for $\omega > 0$ and looking for intersection points gives an infinite family of solutions: call these $\omega_1, \omega_2, \ldots$, with corresponding eigenvalues $\lambda_n = \omega_n^2$. Representative eigenfunctions are obtained by taking $A = 1$ and $\omega = \omega_n$ in (†):

$$y_n(x) = \cos(\omega_n x), \quad \omega_n \text{ given by (†)}, \quad n = 1, 2, \ldots.$$  

Since $L(\omega) \to 0$ as $\omega \to \infty$, the points where $L(\omega) = R(\omega)$ get closer and closer to zeros of $R(o) = \tan \omega$ as $\omega$ increases. These zeros are known: they are the integer multiples of $\pi$. Thinking carefully about the subscripting scheme introduced above (see the sketch) leads to the approximation

$$\omega_n \approx n\pi, \quad \text{so} \quad \lambda_n \approx n^2 \pi^2, \quad n \gg 1.$$  

(c) [Task 1: Change of Basis/Coefficient Extraction] Outline the construction for a general eigenfunction expansion

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x)$$

and then evaluate the coefficients for $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2, \\ 0 & \text{if } 1/2 \leq x \leq 1. \end{cases}$

The general summation formula must involve all the eigenfunctions for the whole problem. Our numbering scheme starts with $n = 0$: $\lambda_0 = -\alpha_0^2 < 0$ while for $n \geq 1$, $\lambda_n = \omega_n^2 > 0$. So in our case the eigenfunction expansion looks like

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x) = c_0 y_0(x) + c_1 y_1(x) + c_2 y_2(x) + \cdots$$

$$= c_0 \left[e^{\alpha_0 x} + e^{-\alpha_0 x}\right] + c_1 \cos(\omega_1 x) + c_2 \cos(\omega_2 x) + \cdots.$$  

For any fixed $k$, multiplying across the given identity by $y_k(x)r(x) = y_k(x)$ and integrating the result lets us harness the power of the orthogonality relation in part (a):

$$\int_0^1 f(x)y_k(x)r(x) \, dx = \sum_{n=0}^{\infty} c_n \int_0^1 y_n(x)y_k(x)r(x) \, dx = c_k \int_0^1 y_k(x)^2 r(x) \, dx$$

$$\implies \quad c_k = \frac{\int_0^1 f(x)y_k(x)r(x) \, dx}{\int_0^1 y_k(x)^2 r(x) \, dx}, \quad k = 0, 1, 2, \ldots$$

The boxed formula works in any Sturm-Liouville problem; here, we have $r(x) = 1$. To match the subscripts in the original expansion above, copy the coefficient formula with $k = n$:

$$c_n = \frac{\int_0^1 f(x)y_n(x) \, dx}{\int_0^1 y_n(x)^2 \, dx}, \quad n = 0, 1, 2, \ldots.$$
Since each $y_n$ is a known function, the denominators here can be evaluated with no reference to whatever function $f(x)$ we happen to be considering:

$$\int_0^1 y_0(x)^2 \, dx = \int_0^1 \left[ e^{2\alpha_0 x} + 2 + e^{-2\alpha_0 x} \right] \, dx = \frac{e^{2\alpha_0} - 1}{2\alpha_0} + 2 - \frac{e^{-2\alpha_0} - 1}{2\alpha_0},$$

$$n \geq 1 \implies \int_0^1 y_n(x)^2 \, dx = \int_0^1 \cos^2(\omega_n x) \, dx = \frac{1}{2} + \frac{\sin(2\omega_n)}{4\omega_n}.$$ 

(In contrast to the famous Big Four eigenvalue expansions, the value of the denominator integral now depends on the counter, $n$.) For the specific $f$ in question, the numerators can be evaluated by splitting:

$$\int_0^1 f(x)y_0(x) \, dx = \int_0^{1/2} \left[ e^{\alpha_0 x} + e^{-\alpha_0 x} \right] \, dx = \frac{e^{\alpha_0/2} - 1}{\alpha_0} - \frac{e^{-\alpha_0/2} - 1}{\alpha_0},$$

$$\int_0^1 f(x)y_n(x) \, dx = \int_0^{1/2} \cos(\omega_n x) \, dx = \frac{\sin(\omega_n/2)}{\omega_n}, \quad n = 1, 2, \ldots .$$

Combining numerators and denominators gives the following results. They are not pretty, but they are completely explicit—recall that $\alpha_0 = 1.1997$, $\omega_1 = 2.7984$, $\omega_2 = 6.1213$, etc., are known constants.

$$c_0 = \frac{\alpha_0^{-1} \left[ e^{\alpha_0/2} - e^{-\alpha_0/2} \right]}{2 + \frac{1}{2} \alpha_0^{-1} \left[ e^{2\alpha_0} - e^{-2\alpha_0} \right]} = \frac{2 \left[ e^{\alpha_0/2} - e^{-\alpha_0/2} \right]}{4\alpha_0 + e^{2\alpha_0} - e^{-2\alpha_0}},$$

$$c_n = \frac{\omega_n^{-1} \sin(\omega_n/2)}{(1/2) + (1/4) \omega_n^{-1} \sin(2\omega_n)} = \frac{4 \sin(\omega_n/2)}{2\omega_n + \sin(2\omega_n)}, \quad n = 1, 2, \ldots .$$

(d) [Task 2: Equation-Solving] Use a suitable eigenfunction series to solve

$$y'' - 17y = f(x), \quad 0 < x < 1; \quad y(0) = 0 = y(1) - y'(1),$$

with $f$ as above.

Here it’s critical that the BC’s and differential operator for $y$ have the same form as the BC’s and differential operator in the original eigenvalue problem. This makes it safe to postulate a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} K_n y_n(x), \quad (*)$$

using the eigenfunctions found above, because any choice of constants $K_n$ will respect the BC’s. Finding the constants $K_n$ that also respects the given ODE will solve the problem. Notice that $y_n''(x) = -\lambda_n y_n(x)$ for each $n \geq 0$, so using $(*)$ in the given equation gives

$$f(x) = y'' - 17y = \sum_{n=0}^{\infty} \left[ K_n y_n''(x) - 17 K_n y_n(x) \right]$$

$$= \sum_{n=0}^{\infty} \left[ -\lambda_n - 17 \right] K_n y_n(x),$$

This is an eigenfunction expansion for $f$. We know such expansions must be unique, and we already have one in hand from part (c) above. This forces

$$- K_0(\lambda_0 + 17) = c_0, \quad \text{i.e.,} \quad K_0 = -\frac{c_0}{\lambda_0 + 17} = -\frac{c_0}{17 - \alpha_0^2},$$

$$- K_n(\lambda_0 + 17) = c_n, \quad \text{i.e.,} \quad K_n = -\frac{c_n}{\lambda_n + 17} = -\frac{c_0}{17 + \omega_n^2}, \quad n = 1, 2, 3, \ldots,$$

with $c_0, c_1, \ldots$ as found in part (c) above.

(e) [Task 3: Dynamics–Wave Type] Solve this wave-motion problem, using $f(x)$ as given above:

(PDE) \quad u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0,

(BC) \quad u_x(0, t) = 0, \quad u(1, t) - u_x(1, t) = 0, \quad t > 0,

(IC) \quad u(x, 0) = f(x), \quad 0 < x < 1,

(IC) \quad u_t(x, 0) = 0, \quad 0 < x < 1.

Eigen-analysis: Since PDE/BC are both homogeneous, we try separation of variables. Putting $u(x, t) = X(x)T(t)$ into PDE/BC produces an eigenvalue problem for $X(x)$ of exactly the form shown above.

Postulate: So we postulate a series solution of the form

$$u(x, t) = \sum_{n=0}^{\infty} T_n(t)y_n(x),$$

using the eigenfunctions already identified. Here the functions $T_0(t), T_1(t), \ldots$ are not yet known: finding them will complete the solution.

Initialize: The initial conditions require $f(x) = u(x, 0)$, so $T_n(0) = c_n$ as calculated in part (c). Also, $0 = u_t(x, 0)$ gives $T_n(0) = 0$ for each $n$.

Propagate: Plugging the postulated series into the PDE produces a differential equation for each coefficient function.

$$\ddot{T}_n(t) + c^2\lambda_n T_n(t) = 0, \quad n = 0, 1, 2, \ldots.$$  

For $n = 0$, this will be unstable because $\lambda_0 = -\alpha_0^2 < 0$: using the initial information found above gives

$$\ddot{T}_0(t) - c^2\alpha_0^2 T_0(t) = 0 \implies T_0(t) = c_0 \left[ e^{\alpha_0 ct} + e^{-\alpha_0 ct} \right]/2.$$  

For $n \geq 1$, this will be a harmonic oscillator equation because $\lambda_n = \omega_n^2 > 0$. Our initial data lead to

$$\ddot{T}_n(t) + c^2\omega_n^2 T_n(t) = 0 \implies T_n(t) = c_n \cos(\omega_n ct).$$

Report: The full solution will be

$$u(x, t) = \frac{c_0}{2} \left[ e^{\alpha_0 ct} + e^{-\alpha_0 ct} \right] \left[ e^{\alpha_0 x} + e^{-\alpha_0 x} \right] + \sum_{n=1}^{\infty} c_n \cos(\omega_n ct) \cos(\omega_n x),$$

with the constants $c_0, c_1, \ldots$ appropriate for $f$ coming from part (c) above.

[Nonhomogeneous heat example not shown.]
D. Extended Example 3

Boyce/DiPrima (7/e) theme example: 11.1 #8; 11.2 #4, 11; 11.3 #4.

Discuss:
\[
\begin{align*}
(\text{ODE}) \quad y'' + \lambda y &= 0, \quad 0 < x < 1, \\
(\text{BC}(0)) \quad y'(0) &= 0, \\
(\text{BC}(1)) \quad y(1) + y'(1) &= 0.
\end{align*}
\]

Solution. All eigenvalues are positive. To see this, first rule out \( \lambda = 0 \) by solving \( y''(x) = 0 \) and applying the BC’s to get the unique solution \( y(x) = 0 \). Then imagine that some eigenvalue-eigenfunction pair \( \lambda, y \) satisfies all the conditions above. Multiply through (ODE) by \( y \), rearrange, and integrate by parts to get
\[
\lambda \int_0^1 y(x)^2 \, dx = -\int_0^1 y(x)y''(x) \, dx = - \left[ y(x)y'(x) \right]_0^1 - \int_0^1 (y'(x))^2 \, dx.
\]
Plug in \( y'(0) = 0 \) and \( y'(1) = -y(1) \) to arrive at
\[
\lambda \int_0^1 y(x)^2 \, dx = y(1)^2 + \int_0^1 (y'(x))^2 \, dx.
\]
Clearly this requires \( \lambda \geq 0 \), and we know \( \lambda \neq 0 \) already. So let \( \omega = \sqrt{\lambda} \), etc., and find that \( y_n(x) = \cos(\omega_n x) \) is an eigenfunction for \( \lambda_n \) if and only if \( \omega_n \) is the \( n \)th positive solution of
\[
\frac{1}{\omega} = \tan \omega.
\]
Sketching the functions of \( \omega \) in this equation helps us predict that \( \omega_1 \approx \pi/4, \omega_2 \in (\pi, 3\pi/2) \), and \( \omega_n \approx (n - 1)\pi \) for \( n \gg 1 \).

![Graph of \( \frac{1}{\omega} = \tan \omega \) and \( \omega_n \) identified above.](image)

Eigenfunction Expansion. Problem: Find constants \( c_n \) such that
\[
x = \sum_{n=1}^\infty c_n \cos(\omega_n x), \quad 0 < x < 1,
\]
for the constants \( \omega_n \) identified above.
Solution. The orthogonality of the eigenfunctions $y_n = \cos(\omega_n x)$ is critical here. The usual derivation, based on

$$0 = \int_0^1 y_m(x)y_n(x)\,dx = \int_0^1 \cos(\omega_m x)\cos(\omega_n x)\,dx \quad \text{whenever } m \neq n,$$

leads to

$$c_m = \frac{\int_0^1 f(x)y_m(x)\,dx}{\int_0^1 y_m(x)^2\,dx}.$$ 

Standard integration by parts will give

$$\begin{align*}
\text{top} &= \int_0^1 x\cos(\omega_m x)\,dx = \left. \frac{\omega_k \sin \omega_k + \cos \omega_k - 1}{\omega_k^2} \right|, \\
\text{bot} &= \int_0^1 \cos^2(\omega_m x)\,dx = \int_0^1 \frac{1 + \cos(2\omega_m x)}{2}\,dx = \left. \frac{\omega_m + \sin(\omega_m)\cos(\omega_m)}{2\omega_m} \right|.
\end{align*}$$

Thus we have

$$c_m = \left. \frac{2}{\omega} \cos \omega + \omega \sin \omega - 1}{\omega + \sin \omega \cos \omega} \right|_{\omega=\omega_m}.$$

The textbook answer (B/DiP, 11.3 #11) says

$$x = \sum_{m=1}^{\infty} a_m \sqrt{\frac{2}{1 + \sin^2 \omega_m}} \cos(\omega_m x), \quad \text{where} \quad a_m = \sqrt{\frac{2}{1 + \sin^2 \omega_m}} \left( \frac{2\cos \omega_m - 1}{\omega_m^2} \right).$$

This suggests an expansion similar to ours, but with coefficients

$$a_m \sqrt{\frac{2}{1 + \sin^2 \omega_m}} = \left[ \frac{2 (2\cos \omega - 1)}{\omega^2 (1 + \sin^2 \omega)} \right]_{\omega=\omega_m}.$$

These are the same values we calculated, even though they look quite different. To explain this, remember that $1/\omega_m = \tan \omega_m$, so $\omega_m \sin \omega_m = \cos \omega_m$ and hence

$$\omega_m + \sin \omega_m \cos \omega_m = \omega_m (1 + \sin^2 \omega_m).$$

PDE BVP. It’s no trouble to invent a partial differential equation problem for which the ingredients above turn out to be key.

\begin{align*}
(PDE) & \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \\
(BC) & \quad u_x(0, t) = 0, \quad u(1, t) + u_x(1, t) = 0, \quad t > 0, \\
(IC) & \quad u(x, 0) = x, \quad 0 < x < 1.
\end{align*}

Separation of variables will show that $u(x, t) = X(x)T(t)$ satisfies (PDE)+(BC) if and only if $X(x)$ happens to be one of the eigenfunctions found above. So we postulate a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \cos(\omega_n x),$$
initialize $T_n(0) = c_n$ as detailed above, and then propagate by plugging the series form into the (PDE). The result:

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\omega_n^2 \alpha^2 t} \cos(\omega_n x), \quad c_n = \left( \frac{2}{\omega} \right) \left| \frac{\cos \omega + \omega \sin \omega - 1}{\omega + \sin \omega \cos \omega} \right|_{\omega = \omega_n}. $$
E. Full Fourier Series (FFS)

**Periodic Functions.** A function \( f : \mathbb{R} \to \mathbb{R} \) is periodic if there is some number \( k \) compatible with an identity of the form
\[
f(x + k) = f(x), \quad x \in \mathbb{R}.
\]
This \( k \) is called the period of \( f \), and the phrase “\( f \) is \( k \)-periodic” succinctly describes the situation.

**Lemma.** Let \( c_0, c_1, c_2 \) be real constants; let \( k > 0 \). For any solution \( y \) of the differential equation
\[
c_2 y'' + c_1 y + c_0 y = 0, \quad x \in \mathbb{R},
\]
the following statements are equivalent:

(i) The function \( y \) is \( k \)-periodic.

(ii) For each and every open interval \((a, b)\) with \( b - a = k \), one has
\[
y(a) = y(b) \quad \text{and} \quad y'(a) = y'(b).
\]

(iii) There exists some open interval \((a, b)\) such that (*) holds.

**Proof.** (i⇒ii) Assume (i). This means that \( y(x+k) = y(x) \) for all real \( x \). Differentiate both sides: \( y'(x+k) = y'(x) \) for all real \( x \). To prove (ii), pick any open interval \((a, b)\) with \( b - a = k \). Then plug \( x = a \) into the identities above, noting that \( a + k = b \):
\[
y(b) = y(a), \quad y'(b) = y'(a).
\]
This proves (ii).

(ii⇒iii) Obvious. [Assume (ii). One of the intervals compatible with the setup in (ii) is \((0, k)\), so we know \( y(0) = y(k) \) and \( y'(0) = y'(k) \). So the interval \((0, k)\) shows that statement (iii) holds.]

(iii⇒i) Assume (iii). That is, assume that some particular values of \( a \) and \( b = a + k \) make line (*) valid. Define \( u(x) = y(x + k) \). This identity implies \( u'(x) = y'(x + k) \) and \( u''(x) = y''(x + k) \). Since the given differential equation about \( y \) is true at every point, we have
\[
c_2 u''(x) + c_1 u'(x) + c_0 u(x) = c_2 y''(x + k) + c_1 y'(x + k) + c_0 y(x + k) = 0.
\]
Also, from (*),
\[
u(a) = y(a + k) = y(b) = y(a), \quad u'(a) = y'(a + k) = y'(b) = y'(a).
\]
This shows that the function \( u \) satisfies the same ODE and IC’s as the function \( y \).

Thanks to the uniqueness theorem for ODE’s, the functions \( u \) and \( y \) must be identical. That is,
\[
y(x + k) = u(x) = y(x), \quad x \in \mathbb{R}.
\]
This proves (i).
**An Eigenvalue Problem.** Given a constant $k > 0$, we seek nontrivial $k$-periodic solutions of

$$y'' + \lambda y = 0, \quad x \in \mathbb{R}.$$

Which constants $\lambda$ work here, and what are the corresponding solutions? Thanks to the Lemma, the property of being $k$-periodic is equivalent to the property of satisfying a pair of repeating BC’s like

$$y(k) = y(0) \quad \text{and} \quad y'(k) = y'(0).$$

Any $x$-values separated by $k$ can be used to set this up, and a symmetric approach turns out to be convenient later. So define $\ell = k/2$ and set up the problem

\begin{align*}
\text{(ODE)} \quad & y'' + \lambda y = 0, \quad x \in \mathbb{R}, \\
\text{(BC)} \quad & y(\ell) = y(-\ell), \quad y'(\ell) = y'(-\ell).
\end{align*}

This problem does not have Sturm-Liouville form, because its boundary conditions relate behaviour at two different endpoints, whereas the Sturm-Liouville theory requires separate BC’s at each end:

$$\begin{cases}
  c_0 y(-\ell) + c_1 y'(-\ell) = 0, \\
  d_0 y(\ell) + d_1 y'(\ell) = 0.
\end{cases}$$

Since (BC) above is not compatible with the Sturm-Liouville setup, we should not be astonished that some of the main conclusions of Sturm-Liouville theory are not available in this case.

**Some S-L Properties Persist.** For the FFS eigenproblem (ODE)+(BC) above,

1. All the eigenvalues are real numbers.
2. There are no negative eigenvalues (review line (6) in Section A, above).
3. Eigenfunctions for distinct eigenvalues are orthogonal on $-\ell < x < \ell$.

[Home Practice: Adapt earlier proofs to justify these statements.]

**Some S-L Properties are Lost.** In particular, the eigenspaces are not all one-dimensional. Let’s find the eigenfunctions in detail.

**Case $\lambda = 0$.** When $\lambda = 0$, all solutions of (ODE)/(BC) are multiples of $y_0(x) = 1$.

Abstract statement: “Eigenspace $E(0)$ is one-dimensional, with basis $\{y_0\}$.”

**Case $\lambda > 0$.** Define $\omega = \sqrt{\lambda} > 0$. Then the general solution of (ODE) is

$$y(x) = A \cos(\omega x) + B \sin(\omega x); \quad \text{note} \quad y'(x) = B\omega \cos(\omega x) - A\omega \sin(\omega x).$$

The BC’s require

$$0 = y(\ell) - y(-\ell) = 2B \sin(\omega \ell),$$

$$0 = y'(\ell) - y'(-\ell) = -2A\omega \sin(\omega \ell).$$
If \( \sin(\omega \ell) \neq 0 \), we get \( A = B = 0 \) and \( y(x) = 0 \) is the trivial solution. However, if \( \sin(\omega \ell) = 0 \), i.e., \( \omega = n\pi/\ell \) for \( n \in \mathbb{N} \), then both BC equations above reduce to “0 = 0” no matter what constants \( A \) and \( B \) we choose. Therefore the positive eigenvalues are precisely

\[
\lambda_n = \omega_n^2 = \frac{n^2\pi^2}{\ell^2}, \quad n = 1, 2, \ldots ,
\]

and for each integer \( n \geq 1 \), each and every nonzero function of the form

\[
y(x) = A \cos \left( \frac{n\pi x}{\ell} \right) + B \sin \left( \frac{n\pi x}{\ell} \right), \quad A, B \in \mathbb{R}
\]
is an eigenfunction for \( \lambda = \lambda_n \). In other words, the set of eigenfunctions for \( \lambda_n \) consists of all (nonzero) linear combinations of the two functions

\[
y_n(x) \overset{\text{def}}{=} \cos \left( \frac{n\pi x}{\ell} \right), \quad \psi_n(x) \overset{\text{def}}{=} \sin \left( \frac{n\pi x}{\ell} \right).
\]

Notice that

\[
(y_n, \psi_n) = \int_{-\ell}^{\ell} \cos \left( \frac{n\pi x}{\ell} \right) \sin \left( \frac{n\pi x}{\ell} \right) \, dx = 0 \quad \text{(odd symmetry)}.
\]

In abstract terminology, we have for each integer \( n \geq 1 \) that the eigenspace \( E(\lambda_n) \) is two-dimensional, with orthogonal basis \( \{y_n, \psi_n\} \).

**Eigenfunction Series.** Given some function \( f(x) \) defined for \(-\ell < x < \ell\), we might hope to for an identity of the form

\[
f(x) = \frac{1}{2} a_0 y_0(x) + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right].
\]

For such an identity to hold, orthogonality requires

\[
b_n = \frac{\int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx}{\int_{-\ell}^{\ell} \sin^2 \left( \frac{n\pi x}{\ell} \right) \, dx} = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 1, 2, \ldots ,
\]

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 0, 1, 2, \ldots .
\]

**Convergence.** Is (*) a valid identity between functions? Mostly. Given \( f(x) \), we can calculate the coefficients \( a_n \) and \( b_n \) from the integrals above, and use the results to define

\[
\tilde{f}(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right].
\]

The resulting function \( \tilde{f} \) will obey
(i) \( \tilde{f}(x) = f(x) \) at each point \( x \in (-\ell, \ell) \) where \( f \) is continuous;

(ii) \( \tilde{f} \) is \( 2\ell \)-periodic;

(iii) \( \tilde{f} \) is jump-averaging at every point.

If the original function \( f \) has properties (ii)–(iii), then \( \tilde{f} \) will recover \( f \) exactly. More generally, properties (i)–(iii) characterize the best approximate reconstruction of \( f \) that we can reasonably expect from a series with the given form.

**Even Functions.** If the given \( f: (-\ell, \ell) \to \mathbb{R} \) happens to be even, the FFS coefficients described above can be expressed as follows:

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx \quad \text{(even symmetry)},
\]

\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx = 0 \quad \text{(odd symmetry)}.\]

Since \( b_n = 0 \) for each \( n \), the FFS of \( f \) involves only cosine terms. Further, by symmetry, the FFS coefficient formula for \( a_n \) matches exactly the FCS formula based in the interval \( 0 < x < \ell \). In short, the FFS and the FCS amount to the same thing for even functions.

**Odd Functions.** If \( f: (-\ell, \ell) \to \mathbb{R} \) happens to be odd, the FFS coefficients described above can be expressed as follows:

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx = 0 \quad \text{(odd symmetry)},
\]

\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx \quad \text{(even symmetry)}.\]

Since \( a_n = 0 \) for each \( n \), the FFS of \( f \) involves only sine terms. Further, by symmetry, the FFS coefficients \( b_n \) are identical with the FSS coefficients calculated using the interval \( 0 < x < \ell \). In short, the FFS and the FSS amount to the same thing for odd functions.

Similar statements apply to the HPSS and HPCS. Thus all four pointwise convergence theorems described above can be derived as applications of the single result for the FFS case. Any result for FFS has immediate applications for all other types.

**Convergence Analysis.** Most textbook presentation of the Fourier theory start with the FFS and then derive the Big Four as consequences. Here we play this development in the reverse order. Suppose \( \ell > 0 \) is given and \( f: (-\ell, \ell) \to \mathbb{R} \). For such an \( f \), define two new functions:

\[ f_{\text{even}}(x) = \frac{1}{2} [f(x) + f(-x)], \quad f_{\text{odd}}(x) = f(x) - f_{\text{even}}(x) = \frac{1}{2} [f(x) - f(-x)]. \]

These new functions are named after their symmetry properties:

\[ f_{\text{even}}(-x) = f_{\text{even}}(x), \quad \text{so } f_{\text{even}} \text{ is an even function}, \]

\[ f_{\text{odd}}(-x) = -f_{\text{odd}}(x), \quad \text{so } f_{\text{odd}} \text{ is an odd function}. \]
Also, it’s easy to check the identity

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x), \quad -\ell < x < \ell.$$  

Intuitively the definitions above split any given $f$ into the sum of an even function and an odd function. And we know a lot about Fourier expansions of functions with symmetry properties.

(a) Since $f_{\text{odd}}$ is odd on $(-\ell, \ell)$, its Fourier Sine Series correctly reproduces it (at continuity points) not just in $(0, \ell)$ but indeed in all of $(-\ell, \ell)$. Also,

$$b_n = \frac{2}{\ell} \int_0^\ell f_{\text{odd}}(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx$$

$$= \frac{1}{\ell} \int_{-\ell}^\ell f_{\text{odd}}(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx$$

$$= \frac{1}{\ell} \int_{-\ell}^\ell [f(x) - f_{\text{even}}(x)] \sin \left( \frac{n\pi x}{\ell} \right) \, dx$$

$$= \frac{1}{\ell} \int_{-\ell}^\ell f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx.$$  

(b) Since $f_{\text{even}}$ is even on $(-\ell, \ell)$, its Fourier Cosine Series correctly reproduces it (at continuity points) not just in $(0, \ell)$ but indeed in all of $(-\ell, \ell)$. Also,

$$a_n = \frac{2}{\ell} \int_0^\ell f_{\text{even}}(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx$$

$$= \frac{1}{\ell} \int_{-\ell}^\ell f_{\text{even}}(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx$$

$$= \frac{1}{\ell} \int_{-\ell}^\ell [f(x) - f_{\text{odd}}(x)] \cos \left( \frac{n\pi x}{\ell} \right) \, dx$$

$$= \frac{1}{\ell} \int_{-\ell}^\ell f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx$$

Using the coefficients calculated above, we have the FCS and FSS reconstructions

$$\tilde{f}_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos \left( \frac{n\pi x}{\ell} \right),$$

$$\tilde{f}_{\text{odd}}(x) = \sum_{n=1}^\infty b_n \sin \left( \frac{n\pi x}{\ell} \right).$$
It follows that for every continuity point $x$ of $f$ in the interval $(-\ell, \ell)$,

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x) = \tilde{f}_{\text{even}}(x) + \tilde{f}_{\text{odd}}(x)$$

$$\implies f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right) = \tilde{f}(x),$$

where

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 1, 2, \ldots.$$

More generally, $\tilde{f}$ is piecewise continuous, $2\pi$-periodic and jump-averaging.

**Useful Miscellany.**

1. **DC Offset.** For integers $n \geq 1$,

$$\int_{-\ell}^{\ell} \cos \left( \frac{n\pi x}{\ell} \right) \, dx = 0 \quad \text{(by calculation, or because $(y_0, y_n) = 0$)}$$

$$\int_{-\ell}^{\ell} \sin \left( \frac{n\pi x}{\ell} \right) \, dx = 0 \quad \text{(by odd symmetry, or because $(y_0, \psi_n) = 0$)}$$

It follows that

$$\int_{-\ell}^{\ell} f(x) \, dx = \int_{-\ell}^{\ell} \left[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right) \right] \, dx = \frac{a_0}{2} 2\ell.$$

That is,

$$\frac{a_0}{2} = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx.$$ 

This shows that the constant term in the FS (or the FCS) reveals the average value of the input function $f$ over one period.

2. **Coefficient Decay.** For any reasonable (i.e., piecewise smooth) function $f$, there is a constant $M$ (independent of $n$) such that both

$$|a_n| \leq \frac{M}{n} \quad \text{and} \quad |b_n| \leq \frac{M}{n} \quad \text{for all } n \geq 1.$$ 

In particular, $a_n \to 0$ and $b_n \to 0$ as $n \to \infty.$
Proof \((a_n \text{ only}; \ b_n \text{ similar}): \) Integrate by parts to get

\[
\ell a_n = \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx
\]

\[
= \left[ f(x) \cdot \left( \frac{\ell}{n\pi} \right) \sin \left( \frac{n\pi x}{\ell} \right) \right]_{-\ell}^{\ell} - \int_{-\ell}^{\ell} \left( \frac{\ell}{n\pi} \right) \sin \left( \frac{n\pi x}{\ell} \right) f'(x) \, dx
\]

\[
\Rightarrow \ell |a_n| = 0 + \frac{\ell}{n\pi} \left| \int_{-\ell}^{\ell} \sin \left( \frac{n\pi x}{\ell} \right) f'(x) \, dx \right|
\]

\[
\leq \frac{\ell}{n\pi} \int_{-\ell}^{\ell} 1 \cdot |f'(x)| \, dx
\]

\[
\Rightarrow |a_n| \leq \frac{1}{n\pi} \int_{-\ell}^{\ell} 1 \cdot |f'(x)| \, dx.
\]

This gives the statement above, with \(M = \frac{1}{\pi} \int_{-\ell}^{\ell} |f'(x)| \, dx.\)