Prerequisites in Logic and Set Theory

A UBC M320 Supplement by Philip D. Loewen

Statements. We're interested in "statements", also known as "logical propositions": these are unambiguous declarative sentences that are either true or false. They could involve symbols as well as words. Examples include the statements a and b below.

$$\begin{aligned} a: &\qquad 2+2=4. \\ b: &\qquad 2+2=5. \end{aligned}$$

Statement a is true; statement b is false. It is possible to string words together and form grammatically correct sentences whose meanings depend on interpretation, or might be both a little bit true and a little bit false all at once. Such constructions are outside the scope of our course.

Sets. Everything comes down to sets. A **set** is a collection of **objects.** We define neither of these terms, relying instead on the following naïve operational understanding. A set is like a **data structure,** with three capabilities:

(i) Simple Lookup: For any object x and set S, exactly one of the statements "$x \in S$" or "$x \notin S$" is true; the other is false. An object x is an *element of* S if and only if $x \in S$.

(ii) Automatic Redundancy Suppression: The sets $A = \{0,1,0,1,0,1,0,1,\ldots\}$ and $B = \{0,1\}$ are identical. (A precise definition of equality between sets is given below.)

(iii) Indexing: Any given set S can release each object it contains in some systematic fashion that makes it possible to assign unambiguous true/false values to definitions or statements of the form, "For every object x in S, ..." [E.g., "For every real number t, define $A_t = \{x \in \mathbb{R} : x \geq t\}$."]

Synonyms. There is no restriction on the types of "object" inside a set. Since a set is itself a type of object, a set can contain other sets. So it's logically possible, and sometimes useful, to think about a "set of sets". It can be more understanable to speak of a "family of sets", or a "collection of sets", or something similar, but in such cases the terms below are all considered equivalent:

$$\text{set} = \text{family} = \text{collection} = \cdots = \text{aggregate} = \text{bag} = \text{sack} = \cdots.$$

The Empty Set. We use $\emptyset$ to denote **the empty set.** Its defining property is that the statement "$x \in \emptyset$" is *false* for every object x. In other words, it is *true* to say "For all x, $x \notin \emptyset$". Now the empty set is an object itself, so it may (or may not) lie inside other sets. But be careful: The set $A = \{\emptyset\}$ is different from the set $\emptyset$: $\emptyset \in A$ is true, whereas $\emptyset \in \emptyset$ is false. The notation $\{\}$ is a plausible replacement for $\emptyset$; Rudin uses the notation 0. We avoid this because the symbol 0 also stands for an important number, and it's confusing to use the same name for two different things.

Russell's Paradox (1901) [optional]. The setup outlined above sounds utterly sensible, but it conceals some logical dangers. To illustrate these, start by classifying sets according to this definition:

$$\begin{aligned} \text{A set } S \text{ is } \quad \textbf{normal} &\Leftrightarrow \; S \notin S; \\ \text{A set } S \text{ is } \textbf{abnormal} &\Leftrightarrow \; S \in S. \end{aligned}$$

Clearly every set is either normal or abnormal, and its impossible for a set to be both at once. Now let N be the collection of all normal sets. What type of set is N?

- If N is normal, then the definition gives $N \notin N$. Being outside the set N means that N is abnormal. But no set can be both normal and abnormal, so this can't happen.
- If N is abnormal, then the definition gives $N \in N$. Being an element of the set N means that N must be normal. But no set can be both normal and abnormal, so this can't happen either.

Something terrible is happening here. The normal/abnormal scheme classifies all sets into two distinct categories, and our N cannot logically belong to either one. What's wrong? Our naïve view of "sets" and "objects"! To be completely solid, we need some fundamental rules about what kinds of collections get to be called "sets", and these rules should disqualify N from that category. For the purposes of MATH 320, we adopt the 9-axiom setup named ZFC, named after Zermelo and Fraenkel plus the Axiom of Choice. Wikipedia has details, on the page entitled "Zermelo-Fraenkel_set_theory". This is the standard axiomatic basis for working mathematicians, and it is largely compatible with the naïve approach. We will stay out of trouble by taking care never to contemplate something as vast and vague as the set of all sets. Let us press on.

Notation for Logic. For logical propositions a and b,

$a \vee b$ means "a or b": it's true exactly when a or b or both are true,

$a \wedge b$ means "a and b": it's true exactly when a and b are true simultaneously,

$\sim a$ means "not a": its truth value is opposite to that of a,

$[a \Rightarrow b]$ means "a implies b": its truth value is $(\sim a) \vee b$. (Think about *rejecting* it.)

$[a \Leftrightarrow b]$ means "a if and only if b": it's true exactly when the truth values of a and b are the same.

Definitions (Sets). For given subsets A and B of some "universal" set X,

$A \cup B \overset{\text{def}}{=} \{x \in X \,:\, (x \in A) \vee (x \in B)\}$ is the *union* of A and B,

$A \cap B \overset{\text{def}}{=} \{x \in X \,:\, (x \in A) \wedge (x \in B)\}$ is the *intersection* of A and B

$A^c \overset{\text{def}}{=} X \setminus A = \{x \in X \,:\, \sim(x \in A)\}$ is the *complement* of A, and

$B \setminus A \overset{\text{def}}{=} B \cap A^c = \{x \in X \,:\, (x \in B) \wedge \sim(x \in A)\}$.

Note the connections between $\cup$ and $\vee$, $\cap$ and $\wedge$, $()^c$, and $\sim$. For sets A and B, the four expressions below have identical meanings:

$$A \subseteq B, \qquad A \subset B, \qquad B \supseteq A, \qquad B \supset A.$$

They all mean this: "$\forall x \in A$, one has $x \in B$"; or, $x \in A \Rightarrow x \in B$. The definition of "$A = B$" is $(A \subseteq B) \wedge (B \subseteq A)$; equivalently, $(x \in A) \Leftrightarrow (x \in B)$.

Tautologies. Imagine doing algebra with logical propositions. A *tautology* is a statement like "$a \vee (\sim a)$" that comes out true for all possible T/F values of the variables involved. These can be useful, especially when the central feature is "$\Leftrightarrow$". Favourites include

$$\begin{aligned} [a \Rightarrow b] &\iff [(\sim b) \Rightarrow (\sim a)] && \text{(contraposition)} \\ [a \Leftrightarrow b] &\iff [a \Rightarrow b] \wedge [b \Rightarrow a] && \text{(equivalence)} \\ \sim(a \wedge b) &\iff (\sim a) \vee (\sim b) && \text{(de Morgan)} \\ \sim(a \vee b) &\iff (\sim a) \wedge (\sim b) && \text{(de Morgan)} \end{aligned}$$

Here are the corresponding statements about sets:

$$\begin{aligned} A \subseteq B &\iff A^c \supseteq B^c \\ A = B &\iff [A \subseteq B] \wedge [B \subseteq A] \\ (A \cap B)^c &= A^c \cup B^c \\ (A \cup B)^c &= A^c \cap B^c. \end{aligned}$$

Quantifiers. Suppose S is a set, and for every object x in S, we have a statement $a(x)$ that involves x.

- $(\forall x \in S)\, a(x)$ means "for each object x in set S, statement $a(x)$ is true". This true automatically when $S = \emptyset$; otherwise $\forall$ works like a generalized form of "ans". [E.g., if $S = \{0, 1, 2\}$, $(\forall x \in S)\, a(x)$ is the same as $a(0) \wedge a(1) \wedge a(2)$.]
- $(\exists x \in S)\, a(x)$ means "there exists some object x in set S for which statement $a(x)$ is true". It's automatically false when $S = \emptyset$; otherwise $\exists$ works like a generalized "or". [E.g., if $S = \{0, 1, 2\}$, $(\exists x \in S)\, a(x)$" says $a(0) \vee a(1) \vee a(2)$.]

Important: The quantifiers $\forall$ and $\exists$ do not commute. If S and T are sets, the two statements below are quite different (think of a specific example!):

(i) $\forall x \in S,\ \exists y \in T\ :\ a(x, y)$

(ii) $\exists y \in T\ :\ \forall x \in S,\ a(x, y)$.

Negations. As de Morgan's laws suggest, and common sense confirms,

$$\begin{aligned} \sim[(\forall x \in S)\, a(x)] &\iff (\exists x \in S)[\sim a(x)] \\ \sim[(\exists x \in S)\, a(x)] &\iff (\forall x \in S)[\sim a(x)]. \end{aligned}$$

Families of Sets. If $\mathcal{A}$ is a set in which every element is itself a set ["a family of sets"], we define the large-scale union and intersection operators as follows:

$$\bigcup \mathcal{A} \stackrel{\text{def}}{=} \bigcup_{A \in \mathcal{A}} A = \{x \in X\ :\ \exists A \in \mathcal{A} \text{ s.t. } x \in A\},$$

$$\bigcap \mathcal{A} \stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{A}} A = \{x \in X\ :\ \forall A \in \mathcal{A},\ x \in A\}.$$

De Morgan's Laws here take the form

$$\left[\bigcup \mathcal{A}\right]^c = \bigcap_{A\in\mathcal{A}} A^c, \qquad \left[\bigcap \mathcal{A}\right]^c = \bigcup_{A\in\mathcal{A}} A^c.$$

When the family $\mathcal{A}$ contains one set for each positive integer, labelled so that $\mathcal{A} = \{A_1, A_2, A_3, \ldots\}$, we mimic classic sigma-notation as follows:

$$\bigcup \mathcal{A} = \bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots,$$
$$\bigcap \mathcal{A} = \bigcap_{i\in\mathbb{N}} A_i = \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots.$$