MATH 320 BACKGROUND

UBC M320 Lecture Notes by Philip D. Loewen

Statements. We're interested in "statements", also known as "logical propositions": these are unambiguous declarative sentences that are either true or false. They could involve symbols as well as words. Examples include the statements a and b below.

$$\begin{aligned} a: &\qquad 2+2=4. \\ b: &\qquad 2+2=5. \end{aligned}$$

Statement a is true; statement b is false. It is possible to concatenate English words together and form sentences that are grammatically correct, but whose meanings depend on interpretation, or might be both a little bit true and a little bit false at the same time. Such constructions are outside the scope of our course.

Sets. Everything comes down to sets. A **set** is a collection of **objects.** We define neither of these terms, relying instead on the following naïve operational understanding. A set is like a **data structure,** with three capabilities:

(i) Simple Lookup: For any object x and set S, exactly one of the statements "$x \in S$" or "$x \notin S$" is true; the other is false. Saying that an object x *is an element of* S is the prose-form equivalent to the symbolic statement $x \in S$.

(ii) Automatic Redundancy Suppression: The sets $A = \{0, 1, 0, 1, 0, 1, 0, 1, \ldots\}$ and $B = \{0, 1\}$ are identical. (A precise definition of equality between sets is given below.)

(iii) Generating/selecting: Any given set S can expose each object it contains in some systematic fashion that makes it possible to assign unambiguous true/false values to statements of the form, "For every object x in S, $\ldots$," and "For some object x in S, $\ldots$." E.g., "For every real number t, define $A_t = \{x \in \mathbb{R} : x \geq t\}$." Or, "For some integer n, there exists an integer k satisfying $n = k^2$."

Synonyms. There is no restriction on the types of "object" inside a set. Since a set is itself a type of object, a set can contain other sets. So it's logically possible, and sometimes useful, to think about a "set of sets". It can be more understandable to speak of a "family of sets", or a "collection of sets", or something similar, but in such cases the terms below are all considered equivalent:

$$\text{set} = \text{family} = \text{collection} = \cdots = \text{aggregate} = \text{bag} = \text{sack} = \cdots.$$

The Empty Set. We use $\emptyset$ to denote **the empty set.** Its defining property is that the statement "$x \in \emptyset$" is *false* for every object x. In other words, it is *true* to say "For all x, $x \notin \emptyset$". Now the empty set is an object itself, so it may (or may not) lie inside other sets. But be careful: The set $A = \{\emptyset\}$ is different from the set $\emptyset$: $\emptyset \in A$ is true, whereas $\emptyset \in \emptyset$ is false. The notation $\{\}$ is a plausible replacement for $\emptyset$; Rudin uses the notation 0. We avoid this because the symbol 0 also stands for an important number, and it's confusing to use the same name for two different things.

Notation for Logic. For logical propositions a and b,

$a \vee b$ means "a or b": it's true exactly when at least one of a or b is true,

$a \wedge b$ means "a and b": it's true exactly when both a and b are true,

$\neg a$ means "not a": its T/F value is opposite to the T/F value of a,

$[a \Rightarrow b]$ means "a implies b": its truth value is $(\neg a) \vee b$. (Think about *rejecting* it.)

$[a \Leftrightarrow b]$ means "a if and only if b": it's true exactly when the truth values of a and b are the same.

Definitions (Sets). For given subsets A and B of some "universal" set X,

$$
\begin{aligned}
&A \cup B \overset{\text{def}}{=} \{x \in X \,:\, (x \in A) \vee (x \in B)\} && \text{is the } \textit{union} \text{ of } A \text{ and } B,\\
&A \cap B \overset{\text{def}}{=} \{x \in X \,:\, (x \in A) \wedge (x \in B)\} && \text{is the } \textit{intersection} \text{ of } A \text{ and } B\\
&A^c \overset{\text{def}}{=} X \setminus A = \{x \in X \,:\, \neg(x \in A)\} && \text{is the } \textit{complement} \text{ of } A, \text{ and}\\
&B \setminus A \overset{\text{def}}{=} B \cap A^c = \{x \in X \,:\, (x \in B) \wedge \neg(x \in A)\}\,.
\end{aligned}
$$

Note the connections between $\cup$ and $\vee$; $\cap$ and $\wedge$; $()^c$ and $\neg$. For sets A and B, the four expressions below have identical meanings:

$$A \subseteq B, \qquad A \subset B, \qquad B \supseteq A, \qquad B \supset A.$$

They all mean this: "$\forall x \in A$, one has $x \in B$"; or, $x \in A \Rightarrow x \in B$. The definition of "$A = B$" is $(A \subseteq B) \wedge (B \subseteq A)$; equivalently, $(x \in A) \Leftrightarrow (x \in B)$.

Tautologies. Imagine doing algebra with logical propositions. A *tautology* is a statement like "$a \vee (\neg a)$" that comes out true for all possible T/F values of the variables involved. These can be useful, especially when the central feature is "$\Leftrightarrow$". Favourites include

$$
\begin{aligned}
[a \Rightarrow b] &\iff [(\neg b) \Rightarrow (\neg a)] && \text{(contraposition)}\\
[a \Leftrightarrow b] &\iff [a \Rightarrow b] \wedge [b \Rightarrow a] && \text{(equivalence)}\\
\neg(a \wedge b) &\iff (\neg a) \vee (\neg b) && \text{(de Morgan)}\\
\neg(a \vee b) &\iff (\neg a) \wedge (\neg b) && \text{(de Morgan)}
\end{aligned}
$$

Here are the corresponding statements about sets:

$$
\begin{aligned}
A \subseteq B &\iff A^c \supseteq B^c\\
A = B &\iff [A \subseteq B] \wedge [B \subseteq A]\\
(A \cap B)^c &= A^c \cup B^c\\
(A \cup B)^c &= A^c \cap B^c.
\end{aligned}
$$

Quantifiers and Their Negations. Suppose S is a set, and for every object x in S, we have a statement $a(x)$ in which x may be involved. Here are two compound statements of great relevance.

$(\forall x \in S)\, a(x)$ means "for each object x in the set S, statement $a(x)$ is true". This is true (and uninformative) when $S = \emptyset$; otherwise $\forall$ works like a generalized form of "and".
[E.g., if $S = \{0, 1, 2\}$, "$(\forall x \in S)\, a(x)$" is the same as "$a(0) \wedge a(1) \wedge a(2)$".]

$(\exists x \in S)\, a(x)$ means "there exists some object x in set S for which statement $a(x)$ is true". It's automatically false when $S = \emptyset$; otherwise $\exists$ works like a generalized "or".
[E.g., if $S = \{0, 1, 2\}$, "$(\exists x \in S)\, a(x)$" says "$a(0) \vee a(1) \vee a(2)$".]

Important: The quantifiers $\forall$ and $\exists$ do not commute. If S and T are sets, the two statements below are quite different (think of a specific example!):

(i) $\forall x \in S,\ \exists y \in T \ :\ a(x, y)$

(ii) $\exists y \in T \ :\ \forall x \in S,\ a(x, y)$.

Negations. As de Morgan's laws suggest, and common sense confirms,

$$\begin{aligned} \neg[(\forall x \in S)\, a(x)] &\iff (\exists x \in S)[\neg a(x)] \\ \neg[(\exists x \in S)\, a(x)] &\iff (\forall x \in S)[\neg a(x)]. \end{aligned}$$

Families of Sets. If $\mathcal{A}$ is a set in which every element is itself a set ["a family of sets"], we define the large-scale union and intersection operators as follows (note that the symbols A and $\mathcal{A}$ are different, and that this distinction matters):

$$\begin{aligned} \bigcup \mathcal{A} &\stackrel{\text{def}}{=} \bigcup_{A \in \mathcal{A}} A = \{x \in X \ :\ \exists A \in \mathcal{A} \text{ s.t. } x \in A\}\,, \\ \bigcap \mathcal{A} &\stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{A}} A = \{x \in X \ :\ \forall A \in \mathcal{A},\ x \in A\}\,. \end{aligned}$$

De Morgan's Laws here take the form

$$\left[\bigcup \mathcal{A}\right]^c = \bigcap_{A \in \mathcal{A}} A^c, \qquad \left[\bigcap \mathcal{A}\right]^c = \bigcup_{A \in \mathcal{A}} A^c.$$

When the family $\mathcal{A}$ contains one set for each positive integer, labelled so that $\mathcal{A} = \{A_1, A_2, A_3, \ldots\}$, we mimic classic sigma-notation as follows:

$$\begin{aligned} \bigcup \mathcal{A} &= \bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots, \\ \bigcap \mathcal{A} &= \bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots. \end{aligned}$$

Numbers. We will use basic facts about the natural numbers ($\mathbb{N} = \{1, 2, 3, \ldots\}$), the integers ($\mathbb{Z}$), and the rationals ($\mathbb{Q}$) freely. In proofs, it's fully acceptable to assert that every nonempty subset of $\mathbb{N}$ contains number smaller than all the others, or that every nonempty subset of $\mathbb{N}$ with an upper bound contains a largest element. Mathematical induction (weak and strong) is a reliable proof technique. Every positive integer admits a unique prime factorization. Quotients and remainders work as expected in the set of integers. We have for every $n \in \mathbb{N}$ and $a, b \in \mathbb{Q}$ that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \qquad \text{where} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(This identity, known as the Binomial Theorem, remains valid in any commutative ring. After constructing the real numbers ($\mathbb{R}$) and the complex numbers ($\mathbb{C}$), we will allow a and b in those sets, too.)