# II. Numbers and Vectors

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## A. The Real Numbers

**Naïve View–Our Plan for Now.** Use $\mathbb{R}$ with $=,<,|\cdot|,+,-,\times,\div$, as always.

**Serious View (Details Later).** Work hard to construct from the axioms a set $\mathcal{R}$ with special elements $\mathbb{O}$ and $\mathbb{I}$, and a subset $\mathbb{P} \subseteq \mathcal{R}$ ("positive elements"), and mappings $A: \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ ("add"), $M: \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ ("multiply"), for which defining the basic operations above in terms of

$$x + y = A(x, y), \qquad x \cdot y = M(x, y), \qquad x > \mathbb{O} \Leftrightarrow x \in \mathbb{P}$$

produces a consistent setup in which (i) all the familiar rules of arithmetic all work; (ii) there is a subset $\mathcal{Q}$ of $\mathcal{R}$ that is in one-to-one correspondence with the set of rational numbers (and the correspondence respects standard arithmetic); *and* (iii) the extra properties of order-completeness and metric completeness also hold.

**Discussion.** The completeness properties are what make $\mathbb{R}$ so special. Any line of reasoning that doesn't use them explicitly looks the same as a line of reasoning where all the numbers involved are rational. Many useful skills and arguments can be developed in $\mathbb{Q}$ and then lifted to $\mathbb{R}$ with no significant change.

**Theorem (Archimedes).** *In $\mathbb{R}$, the set $\mathbb{N}$ has no upper bound. That is,*

$$\forall r \in \mathbb{R},\ \exists n \in \mathbb{N} \,:\, n > r.$$

*Proof.* It's obvious that $\mathbb{N}$ has no upper bound in $\mathbb{Q}$. Verifying that this important property remains valid in $\mathbb{R}$ requires the completeness property. Details later. ////

**Corollaries.** (a) *For any fixed $\varepsilon > 0$, some $n \in \mathbb{N}$ obeys $1/n < \varepsilon$.*

(b) *Whenever $x, y \in \mathbb{R}$ obey $y - x > 1$, we have $(x, y) \cap \mathbb{Z} \neq \emptyset$.*

(c) *For any $a, b \in \mathbb{R}$ with $a < b$, we have both $(a, b) \cap \mathbb{Q} \neq \emptyset$ and $(a, b) \setminus \mathbb{Q} \neq \emptyset$.*

*Proof.* (a) Apply Archimedes to $r = 1/\varepsilon$ to produce $n \in \mathbb{N}$ s.t. $n > 1/\varepsilon$, i.e., $1/n < \varepsilon$.

(b) Let $S = \{n \in \mathbb{Z} \,:\, n \geq y\}$. By Archimedes, $S \neq \emptyset$; by Fact 1, $\widehat{n} = \min(S)$ exists. Let's show $z = \widehat{n} - 1 \in (x, y)$:

(i) $z < y$: By definition of "min", $\widehat{n} - 1 \notin S$. This means, by definition of $S$, that $z = \widehat{n} - 1 < y$.

(ii) $z > x$: We know $\widehat{n} \in S$. So, by the definition of $S$, $\widehat{n} \geq y > x + 1$. Thus $z = \widehat{n} - 1 > x$.

(c) Given $a < b$, apply (a) to show that some $n \in \mathbb{N}$ obeys $1/n < b - a$. Then $nb - na > 1$, so (b) applies to $x = na$, $y = nb$. That is, there must exist some $m \in \mathbb{Z}$ for which $na < m < nb$, or, $a < \dfrac{m}{n} < b$. Thus $\dfrac{m}{n} \in (a, b) \cap \mathbb{Q}$.

Likewise, if $a < b$ then $\dfrac{a}{\sqrt{2}} < \dfrac{b}{\sqrt{2}}$ so some $q \in \mathbb{Q}$ obeys $\dfrac{a}{\sqrt{2}} < q < \dfrac{b}{\sqrt{2}}$. It follows that $q\sqrt{2} \in (a,b) \setminus \mathbb{Q}$.
Alternatively, the set $(a,b) \cap \mathbb{Q}$ must be countable (it's an infinite subset of the countable set $\mathbb{Q}$), but the set $(a,b)$ is uncountable (it's in one-to-one correspondence with the interval $(0,1)$). It follows that $(a,b) \setminus \mathbb{Q}$ must be uncountable too, which certainly implies that it is not empty. ////

**Trichotomy.** For every real number $x$, exactly one of the following is true:

$$x < 0, \qquad x = 0, \qquad x > 0.$$

By taking $x = b - a$, we deduce that whenever $a, b \in \mathbb{R}$, exactly one of the following is true:

$$a < b, \qquad a = b, \qquad a > b.$$

Now for any $a, b \in \mathbb{R}$, it's rather obvious that

$$a > b \implies \exists \varepsilon > 0 \,:\, a \geq b + \varepsilon.$$

(Indeed, if $a > b$ then $\varepsilon = a - b$ obeys the conclusion.) The contrapositive of this statement is logically equivalent, but occasionally useful:

$$\boxed{\left[\forall \varepsilon > 0, a < b + \varepsilon\right] \implies a \leq b.}$$

It reveals that one way to prove the inequality "$a \leq b$" is to prove a that the relaxed inequality $a \leq b + \varepsilon$ actually holds for every fixed number $\varepsilon > 0$.

## B. Finite Dimensional Euclidean Spaces

For any $k \in \mathbb{N}$, we write $\mathbb{R}^k$ for the set of ordered $k$-tuples

$$\mathbf{x} = (x_1, x_2, \ldots, x_k), \quad x_j \in \mathbb{R}.$$

The standard operations on this set are vector addition

$$(x_1, x_2, \ldots, x_k) + (y_1, y_2, \ldots, y_k) = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k),$$

multiplication by real scalars

$$\alpha\,(x_1, x_2, \ldots, x_k) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_k),$$

and the dot product:

$$(x_1, x_2, \ldots, x_k) \bullet (y_1, y_2, \ldots, y_k) = x_1 y_1 + x_2 y_2 + \ldots + x_k y_k = \sum_{j=1}^{k} x_j y_j.$$

The dot product induces the "norm"

$$|\mathbf{x}| = \sqrt{\mathbf{x} \bullet \mathbf{x}} = \sqrt{\sum_{j=1}^{k} x_j^2},$$

which makes $\mathbb{R}^k$ into **Euclidean $k$-space**. (This definition is fully compatible with the usual absolute value on $\mathbb{R} = \mathbb{R}^1$, since $|x| = \sqrt{x^2}$ holds for each real number $x$.) Key algebraic properties are listed in Theorem 1.37: highlights are

(i) $|\mathbf{x}|^2 = \mathbf{x} \bullet \mathbf{x}$ (often useful in proofs),

(ii) the Schwarz inequality

$$|\mathbf{x} \bullet \mathbf{y}| \le |\mathbf{x}|\,|\mathbf{y}|, \quad \text{with "=" iff } \alpha\mathbf{x} + \beta\mathbf{y} = 0 \text{ for some real } \alpha, \beta \text{ not both zero,}$$

(iii) the triangle inequalities

$$|\mathbf{x} + \mathbf{y}| \le |\mathbf{x}| + |\mathbf{y}|, \qquad |\mathbf{x} - \mathbf{y}| \ge \Big| |\mathbf{x}| - |\mathbf{y}| \Big|.$$

**Discussion.** (i) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ and $t \in \mathbb{R}$,

$$\begin{aligned} 0 \le |\mathbf{x} - t\mathbf{y}|^2 &= (\mathbf{x} - t\mathbf{y}) \bullet (\mathbf{x} - t\mathbf{y}) \\ &= \mathbf{x} \bullet \mathbf{x} - t\mathbf{y} \bullet \mathbf{x} - t\mathbf{x} \bullet \mathbf{y} + t^2 \mathbf{y} \bullet \mathbf{y} \\ &= |\mathbf{x}|^2 - 2t\mathbf{x} \bullet \mathbf{y} + t^2 |\mathbf{y}|^2. \end{aligned}$$

If $\mathbf{y} \ne \mathbf{0}$, the right-hand side is a quadratic polynomial in $t$ with at most one real root. Hence, its discriminant cannot be positive:

$$\begin{gathered} (-2\mathbf{x} \bullet \mathbf{y})^2 - 4\left(|\mathbf{x}|^2\right)(|\mathbf{y}|)^2 \le 0, \\ |\mathbf{x} \bullet \mathbf{y}|^2 \le (|\mathbf{x}|\,|\mathbf{y}|)^2. \end{gathered}$$

Taking square roots of both sides gives the Schwarz inequality. (If $\mathbf{y} = \mathbf{0}$, the inequality is obvious.)

(ii) In the calculation above, we may extract

$$|\mathbf{x} - t\mathbf{y}|^2 = |\mathbf{x}|^2 - 2t\mathbf{x} \bullet \mathbf{y} + t^2 |\mathbf{y}|^2, \qquad \text{i.e.,} \qquad |\mathbf{x} - t\mathbf{y}|^2 - t^2 |\mathbf{y}|^2 = \mathbf{x} \bullet \mathbf{x} - 2t\mathbf{x} \bullet \mathbf{y}.$$

Substituting $c = -2t$ and gives

$$\mathbf{x} \bullet \mathbf{x} + c\mathbf{y} \bullet \mathbf{x} = \left| \mathbf{x} + \left(\frac{c}{2}\right)\mathbf{y} \right|^2 - \left(\frac{c}{2}\right)^2 |\mathbf{y}|^2.$$

This extends the algebraic idea of "completing the square" to scalar-valued quadratic functions with a vector variable $\mathbf{x}$.

(iii) The triangle inequality follows from the Schwarz inequality: for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$,

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}) \bullet (\mathbf{x}+\mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \bullet \mathbf{y} + |\mathbf{y}|^2 \\ &\le |\mathbf{x}|^2 + 2|\mathbf{x}|\,|\mathbf{y}| + |\mathbf{y}|^2 \qquad \text{(by Schwarz, since } p \le |p| \ \forall p \in \mathbb{R}) \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2 \\ &= \Big| |\mathbf{x}| + |\mathbf{y}| \Big|^2. \end{aligned}$$

Taking square roots of both sides gives the desired result. ////

*This would be one great place to discuss order-completeness in some detail, but another great place comes up after a first look at limits. Please be patient.*