

### III. Order Completeness

UBC M320 Lecture Notes by Philip D. Loewen

#### A. Supremum and Infimum

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**Definition.** Let  $S \subseteq \mathbb{R}$  be nonempty.

- (a) A number  $r \in \mathbb{R}$  is a *lower bound* for  $S$  when  $\forall s \in S, r \leq s$ .
- (b) A number  $R \in \mathbb{R}$  is an *upper bound* for  $S$  when  $\forall s \in S, s \leq R$ .
- (c) The set  $S$  is *bounded [from] below* if there exists a lower bound for  $S$ .
- (d) The set  $S$  is *bounded [from] above* if there exists an upper bound for  $S$ .
- (e) The set  $S$  is *bounded* if  $S$  is bounded from below and bounded from above.

Here is an extremely important element in our construction of  $\mathbb{R}$ .

#### Order Completeness, a.k.a. the Least Upper Bound Property.

For any nonempty set  $S \subseteq \mathbb{R}$  that has an upper bound, there exists a real number  $\beta$  such that

- (i)  $\forall x \in S, x \leq \beta$   
(i.e.,  $\beta$  is an *upper bound* for  $S$ ); and
- (ii)  $\forall \gamma < \beta, \exists x \in S : \gamma < x$   
(i.e., every  $\gamma$  less than  $\beta$  is too small to be an upper bound for  $S$ —which makes  $\beta$  *the least* upper bound).

This real number is unique; we denote it  $\sup(S)$ . (Alternative:  $\text{lub}(S)$ .)

The phrase “least upper bound” is apt, as the next result emphasizes.

**Proposition.** Suppose  $S \subseteq \mathbb{R}$  is nonempty and  $b$  is an upper bound for  $S$ . Then  $\sup(S) \leq b$ .

*Proof.* Pick any  $\gamma < \sup(S)$ . By definition of supremum, there exists  $x \in S$  such that  $x > \gamma$ . And since  $b$  is an upper bound for  $S$ , this implies  $b > \gamma$ . Since  $\gamma$  is arbitrary, we must have  $b \geq \sup(S)$ , as required.

[Alternative: Pick any  $\varepsilon > 0$ . By definition of supremum, there exists  $x \in S$  such that  $x > \sup(S) - \varepsilon$ . This implies

$$b > \sup(S) - \varepsilon, \text{ i.e., } b - \sup(S) > -\varepsilon.$$

Since this works for every  $\varepsilon > 0$ ,  $b - \sup(S)$  cannot be negative. In other words,  $b \geq \sup(S)$ , as required.] ////

**Supremum vs Maximum.** The generic notation  $\max(S)$  refers to *the element  $\hat{s}$  of the set  $S$*  (if any) that obeys  $\hat{s} \geq s$  for all  $s \in S$ . For many sets  $S \subseteq \mathbb{R}$ , like  $S = (0, 1)$ , no such element exists. However, the numbers  $\inf(S) = 0$  and  $\sup(S) = 1$  are defined. Thus the infimum and supremum provide conceptually robust replacements

for the more fragile ideas of minimum and maximum. We say “the supremum of  $S$  is attained (or achieved)” when  $\sup(S) \in S$ : in every situation where the number  $\max(S)$  is defined, it equals  $\sup(S)$ .

**Infimum vs Minimum.** The words corresponding to “max” and “sup” that we use to describe lower bounds are “min” and “inf”. As above,  $\min(S)$  can be meaningless, whereas  $\inf(S)$  is defined for any nonempty set  $S$  with a lower bound, and whenever  $\inf(S)$  is attained (i.e.,  $\inf(S) \in S$ ), we have  $\inf(S) = \min(S)$ . The two-part definition of  $\inf(S)$  is worth writing explicitly. Existence follows readily from the Least Upper Bound Property.

**Definition.** For any nonempty  $S \subseteq \mathbb{R}$  that has a lower bound, there exists a unique real number  $\alpha$  such that

- (i)  $\forall x \in S, \alpha \leq x$   
(i.e.,  $\alpha$  is a *lower bound* for  $S$ ); and
- (ii)  $\forall \gamma > \alpha, \exists x \in S : \gamma > x$   
(i.e., every  $\gamma$  larger than  $\alpha$  is too big to be a lower bound for  $S$ —which makes  $\alpha$  *the greatest* lower bound).

This real number is denoted  $\inf(S)$ . (Alternative:  $\text{glb}(S)$ .)

**Discussion/Notation.** The whole point of constructing  $\mathbb{R}$  was to guarantee the *existence* of the quantities  $\inf(S)$  and  $\sup(S)$  *as real numbers*, for any nonempty set  $S$  with suitable boundedness properties. In fact, with some new symbols we can extend these ideas and notation to all of  $\mathcal{P}(\mathbb{R})$ .

**Extended Notation.** Introduce the symbols  $-\infty$  and  $+\infty$  (**not real numbers!!**) and define

$\sup S = +\infty \Leftrightarrow$  the set  $S$  has no upper bound,

$\sup S = -\infty \Leftrightarrow S = \emptyset$ .

Of course symmetry then requires

$\inf S = +\infty \Leftrightarrow S = \emptyset$ ,

$\inf S = -\infty \Leftrightarrow$  the set  $S$  has no lower bound.

This scheme gives a “value” (either numeric or symbolic) to the supremum and infimum for any set  $S \subseteq \mathbb{R}$ . Further, “ $\inf S = -\sup(-S)$ ” holds in this extended interpretation. Let’s check that the two-part characterization of  $\beta = \sup(S)$  offered above is compatible with the extended interpretation.

- Case  $\beta = -\infty$  arises only when  $S = \emptyset$ . In this case assertion (i) is valid because there are no elements  $x$  in  $S$  to falsify the strange-looking condition  $x \leq -\infty$ . And assertion (ii) is not wrong because there are no values  $\gamma < -\infty$  for which one must find an element  $x$  in  $S$ .
- Case  $\beta = +\infty$  trivializes test (i), but reduces line (ii) to the assertion that for every  $\gamma$  in  $\mathbb{R}$ , some element  $x \in S$  has  $x > \gamma$ . Indeed this is equivalent to saying

that  $S$  has no upper bound.

Nuance: a set  $S$  for which  $\sup(S) = +\infty$  has no upper bound, so the phrase “least upper bound” no longer provides a perfectly accurate synonym for “supremum”. (It’s still a helpful mnemonic for structuring proofs, however.)

We also extend the definitions of  $\sup(S)$  and  $\inf(S)$  to cover sets  $S$  in the extended real line  $\mathbb{R} \cup \{\pm\infty\}$ . (See Rudin 1.23, p. 11.) This adds further subtlety to the simple situations above: now the statement that  $\sup(S) = -\infty$  implies only that  $S \subseteq \{-\infty\}$ . (Given the extra information that  $S \subseteq \mathbb{R}$ , we would deduce  $S = \emptyset$ , as before.)

### Low-Hanging Fruit.

- For any set  $S \neq \emptyset$ ,  $\inf(S) \leq \sup(S)$ . (Indeed, any  $s \in S$  obeys both  $\inf(S) \leq s$  and  $s \leq \sup(S)$ . Chain these together.)
- $\inf(\emptyset) = +\infty$  and  $\sup(\emptyset) = -\infty$ . (These reverse the usual inequality most spectacularly.)
- $\inf(S) = \sup(S)$  if and only if  $S$  is a single-element set; that element equals both  $\inf(S)$  and  $\sup(S)$ .
- Whenever  $U, V \subseteq \mathbb{R}$ ,

$$U \subseteq V \implies \inf(V) \leq \inf(U) \text{ and } \sup(U) \leq \sup(V).$$

This is obvious:  $\inf(V)$  is a lower bound for  $V$ , so set inclusion makes it a lower bound for the subset  $U$ . And  $\inf(U)$  is the *greatest* lower bound for  $U$ , so  $\inf(U) \geq \inf(V)$ . Etc. Note that the converse is most certainly false: let  $U = \{-\frac{1}{2}, \frac{1}{2}\}$  and  $V = \{-1, 1\}$  to see a situation where  $\inf(V) < \inf(U)$  and  $\sup(U) < \sup(V)$ , but  $U \not\subseteq V$ .

### B. Archimedes and the Density of the Rationals

**Theorem (Archimedes).** *The set  $\mathbb{N}$  has no upper bound in  $\mathbb{R}$ . In particular, for every  $x \in \mathbb{R}$  there exists  $n \in \mathbb{N}$  satisfying  $n > x$ .*

*Proof.* Suppose, seeking a contradiction, that  $\mathbb{N}$  has an upper bound. Let  $\beta \stackrel{\text{def}}{=} \sup(\mathbb{N})$ . Then  $\beta - \frac{1}{2}$  is not an upper bound for  $\mathbb{N}$ , so there exists  $n \in \mathbb{N}$  such that  $n > \beta - \frac{1}{2}$ . But then  $n + 1$  is a natural number satisfying  $n + 1 > \beta + \frac{1}{2} > \beta$ , and this contradicts the defining property of  $\beta$ . ////

*Remark.* The Archimedean property of  $\mathbb{Q}$  was never in doubt. But not every ordered field containing a work-alike copy of  $\mathbb{Q}$  inherits the property. See Rechnitzer’s M319 notes for a reasonably simple example in which the elements are rational functions with rational coefficients. (<https://arechnitzer.gitlab.io/m319text/>, Example 3.1.5.)

**Definition.** Given any  $a, b \in \mathbb{R}$  with  $a \leq b$ , define these four subsets of  $\mathbb{R}$ :

$$\begin{aligned}(a, b) &= \{x \in \mathbb{R} : a < x < b\}, \\ [a, b) &= \{x \in \mathbb{R} : a \leq x < b\}, \\ (a, b] &= \{x \in \mathbb{R} : a < x \leq b\}, \\ [a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\}.\end{aligned}$$

Extend these definitions to allow  $a = -\infty$  and  $b = +\infty$ , but insist that these appear with round parentheses because neither of  $\pm\infty$  is a real number.

**Proposition.** Every nonempty interval  $(a, b)$  in  $\mathbb{R}$  contains a rational number.

*Proof.* WLOG, assume both  $a, b \in \mathbb{R}$ . Appeal to Archimedes to find an integer  $N > 1/(b - a)$ . Then  $b - 1/N > a$ , so the interval  $(a, b)$  must contain a rational number of the form  $k/N$  for some  $k$ . This is obvious enough, but a full justification might start by choosing the largest integer  $k$  for which  $k < Nb$ . Then  $k/N < b$ , while  $k + 1 \geq Nb$  implies

$$a < b - \frac{1}{N} \leq \frac{k+1}{N} - \frac{1}{N} = \frac{k}{N}.$$

Combine these inequalities to reveal  $\frac{k}{N} \in (a, b)$ , as required. /////

### C. Nested Intervals

**Exercise.** (a) Given a sequence of nonempty closed intervals  $I_1 = [a_1, b_1]$ ,  $I_2 = [a_2, b_2]$ ,  $\dots$ , in  $\mathbb{R}$ , such that  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ , prove that

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset.$$

That is, show that there is a real number  $x$  which belongs to every  $I_n$ .

(Note: The symbol  $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$  is reserved for the case where both  $a$  and  $b$  are real numbers. To encode  $\{t \in \mathbb{R} : t \geq 0\}$ , for example, we would write  $[0, +\infty)$ , not  $[0, +\infty]$ .)

(b) Show that statement (a) can be false if open intervals are used instead of closed intervals.