

IV. Cardinality

UBC M320 Lecture Notes by Philip D. Loewen

Countable sets

2026-09-23

Recall $\mathbb{N} = \{1, 2, 3, \dots\}$, the set of **natural numbers**.

Definition. A set S is **finite** if either $S = \emptyset$ or there exists some $n \in \mathbb{N}$ and some bijection $\phi: \{1, 2, \dots, n\} \rightarrow S$. [Notation: $|\emptyset| = 0$, $|\{1, 2, \dots, n\}| = n$.]

Definition. A set S is **countable** if there exists some bijection $\phi: \mathbb{N} \rightarrow S$. [Notation: $|S| = \aleph_0$.]

Sidebar. • It's regrettable that in this definition, the sets that are not "countable" include all finite sets. Blame Rudin and continue.

- If every infinite set were countable, we wouldn't need to invent a special term. It will be interesting to explore some sets that lack this property.

Example. Some countable sets:

- (a) $\mathbb{N}$ itself (the prototype). Use $\phi(n) = n$, the identity mapping on $\mathbb{N}$.
- (b) Hilbert's Hotel ("shift"): $\mathbb{N} \cup \{0\}$. (Use $\phi(n) = n - 1$.)
- (c) $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. "Shuffle": use $\psi: \mathbb{N} \rightarrow \mathbb{Z}$ defined by

$$\psi(n) = \begin{cases} -n/2, & \text{if } n \text{ is even, i.e., } n = 2k \exists k \in \mathbb{N}, \\ (n-1)/2, & \text{if } n \text{ is odd, i.e., } n = 2k-1 \exists k \in \mathbb{N}. \end{cases}$$

- (d) $\mathbb{N} \times \mathbb{N}$.

Proof. "Diagonalize": Enumerate the ordered pairs in $\mathbb{N} \times \mathbb{N}$ as indicated below:

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Note that diagonal number k includes all pairs whose elements sum to $k + 1$. This makes it easy to show that the enumeration scheme is onto: the pair (m, n) shows up on diagonal number $m + n - 1$, so it's easy to calculate its index in the sequence described above. ////

Countability and Sequences. If A is countable, with $\phi: \mathbb{N} \rightarrow A$ bijective, defining $a_n = \phi(n)$ lets us write $A = \{a_1, a_2, a_3, \dots\}$. Conversely, if $A = \{a_1, a_2, a_3, \dots\}$ is the collection of values from some sequence with distinct terms, then defining $\psi(n) = a_n$ produces a bijection and shows A is countable.

Well-Ordering Property for $\mathbb{N}$. Every nonempty subset S of $\mathbb{N}$ contains a smallest element. In symbols,

$$S \subseteq \mathbb{N}, S \neq \emptyset \implies \exists \hat{s} \in S : \forall s \in S, \hat{s} \leq s. \quad (*)$$

We take this as known; it's the basis for Mathematical Induction. Typical notation for $\hat{s}$ is $\min S$ or $\min(S)$.

Proposition. *Every subset of $\mathbb{N}$ is either finite or countable.*

Proof. Let $A \subseteq \mathbb{N}$. If A is finite, we're done, so assume A is infinite. Write $A_1 = A$. Clearly $A_1 \neq \emptyset$, so let $a_1 = \min(A_1)$. Clearly $a_1 \geq 1$.

Then let $A_2 = A \setminus \{a_1\}$: Since A is infinite, $A_2 \neq \emptyset$, so let $a_2 = \min(A_2)$. Notice $a_2 \geq 1 + a_1$, so $a_2 \geq 2$.

Continue inductively. Having chosen a_n , let $A_{n+1} = A \setminus \{a_1, a_2, \dots, a_n\}$. Since A is infinite, $A_{n+1} \neq \emptyset$, so pick $a_{n+1} = \min A_{n+1}$. Notice $a_{n+1} \geq 1 + a_n$, so $a_{n+1} \geq n + 1$. By induction, this defines a_n for each $n \in \mathbb{N}$. Choosing the smallest available element at each stage guarantees that any number k in A will be selected and labelled in one of the first k steps, i.e., $k = a_j$ for some $j \in \{1, 2, \dots, k\}$. Further, the construction guarantees that the sequence is increasing, i.e.,

$$a_1 < a_2 < a_3 < \dots$$

Now define $\phi: \mathbb{N} \rightarrow A$ by $\phi(n) = a_n$. Let's prove that ϕ is a bijection.

- ϕ is 1-1: If $m \neq n$, then $\phi(m) \neq \phi(n)$ because the sequence $(a_n)_n$ is increasing.
- ϕ is onto: For any $k \in A$, as noted above, we have $k = \phi(j)$ for some $j \leq k$.

This confirms the definition of countability for the set A . /////

2026-09-25

Proposition. *Every subset of any countable set is either finite or countable.*

Proof. Let X be a countable set, with subset S . Then there must be some bijection $\psi: \mathbb{N} \rightarrow X$. Define $f: S \rightarrow \mathbb{N}$ like this

$$f(s) = \psi^{-1}(s), \quad s \in S.$$

Then f is a bijection between S and $f(S)$, a subset of $\mathbb{N}$. By above, $f(S)$ is either finite or countable. Thus the same holds for S . /////

Proposition. *Every infinite set contains a countably infinite subset.*

Proof. Let A be an infinite set; define $A_1 = A$. Choose $a_1 \in A_1$ and let $A_2 = A_1 \setminus \{a_1\}$. Then $A_2 \neq \emptyset$, so we can choose $a_2 \in A_2$ and define $A_3 = A_2 \setminus \{a_2\}$. Continue. An elementary induction argument (omitted) shows that for each $k \geq 2$, a_k is defined in A , and distinct from $a_1, \dots, a_{k-1}$. It follows that $\{a_1, a_2, \dots\}$ is a countable subset of A . (This proof is very similar to the one given above for $A \subseteq \mathbb{N}$, only here we have no mechanism to guarantee that the sequence we build includes every element of A .)
 ////

Remark. For subsets of $\mathbb{N}$, the well-ordering property justifies the process used earlier. But for a generic infinite set, the process of picking a sequence requires some form of the Axiom of Choice (AC). It turns out that AC has several variants, with varying degrees of potency. Here it suffices to appeal to the so-called Axiom of Dependent Choice (DC). Further details are beyond the scope of this course.

Terminology. “Finite or countable” $\equiv$ “at most countable”.

Theorem (Proving Countability). *Given a set A , either (a) or (b) below implies that A is finite-or-countable:*

- (a) *There exist a countable set X and a **one-to-one** mapping $f: A \rightarrow X$,*
- (b) *There exist a countable set X and an **onto** mapping $g: X \rightarrow A$.*

Proof. (a) Write $S = f(A) = \{f(a) : a \in A\}$. Since $S \subseteq X$, S is either finite or countable. Since $f: A \rightarrow S$ is a bijection, the same is true for A .

- (b) Enumerate $X = \{x_1, x_2, \dots\}$. For each a , let $f(a) = \min \{n \in \mathbb{N} : g(x_n) = a\}$. (That set is nonempty because g is onto.) This defines a mapping $f: A \rightarrow \mathbb{N}$ that establishes a one-to-one correspondence between A and $f(A)$. Since $f(A)$ is a subset of $\mathbb{N}$, it must be finite or countable; the same therefore holds for A .

Details (1–1): Suppose $a_1, a_2 \in A$ obey $f(a_1) = f(a_2)$. Then, since $g(x_{f(a)}) = a$ for each $a \in A$,

$$a_1 = g(x_{f(a_1)}) = g(x_{f(a_2)}) = a_2.$$

////

In a memorable phrase, *the range of any sequence is finite or countable.*

More Countable Sets. Here we record some useful consequences of the countability of $\mathbb{N} \times \mathbb{N}$. (Each of the sets discussed below is obviously infinite, which our definition requires.)

1. If A and B are countable sets, then $A \times B$ is countable too.

[Write $A = \{a_1, a_2, \dots\}$, $B = \{b_1, b_2, \dots\}$, and define $g: \mathbb{N} \times \mathbb{N} \rightarrow A \times B$ by $g(m, n) = (a_m, b_n)$. This g is ONTO, and $\mathbb{N} \times \mathbb{N}$ is countable, so the result follows from (b) above.]

2. The set $\mathbb{Z} \times \mathbb{N}$ is countable.

[This is an immediate consequence of the previous point; we showed that $\mathbb{Z}$ and $\mathbb{N}$ are countable previously.]

3. The set $\mathbb{Q}$ of rational numbers is countable.

Pf: The mapping $\phi: \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ defined by $\phi(p, q) = p/q$ is clearly onto (and also clearly NOT 1–1), so $\mathbb{Q}$ is countable by part (b) of the theorem above.

4. Let $\mathcal{S}_2$ be the collection (set) of all nonempty subsets of $\mathbb{N}$ with one or two elements. Then $\mathcal{S}_2$ is countable.

Pf: Recall that $\mathbb{N} \times \mathbb{N}$ is countable. Define $g: \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{S}_2$ by $g(m, n) = \{m, n\}$. Clearly g is onto; since $\mathbb{N} \times \mathbb{N}$ is countable, result follows from (b) above. [E.g., $\{5\} = g(5, 5)$; $\{1, 2\} = g(1, 2) = g(2, 1)$; etc. Note that g is not 1–1.]

5. The union of countably many sets, each one countable, is a countable set.

Pf: Suppose $A^{(1)}, A^{(2)}, \dots$ is a list of countable sets. Enumerate each set—for each $m \in \mathbb{N}$, arrange

$$A^{(m)} = \{a_1^{(m)}, a_2^{(m)}, a_3^{(m)}, \dots\}.$$

Then define $g: \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_{m \in \mathbb{N}} A^{(m)}$ by

$$g(m, n) = a_n^{(m)}.$$

This g might not be 1–1 if the sets $A^{(m)}$ overlap, but it's certainly onto. Since $\mathbb{N} \times \mathbb{N}$ is countable, that's enough to show $\bigcup_{m \in \mathbb{N}} A^{(m)}$ is countable. [Again we have used (b) above.]

6. The set $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} = \{(p, q, r) : p, q, r \in \mathbb{N}\}$ is countable. So is $\mathcal{S}_3$, the collection of nonempty subsets of $\mathbb{N}$ with 3 or fewer elements.

Pf: Since both $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ are countable, so [by item 1] is the set $P = (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$. A bijection $f: \mathbb{N}^3 \rightarrow P$ is

$$f(p, q, r) = ((p, q), r).$$

Clearly $g(p, q, r) = \{p, q, r\}$ maps $\mathbb{N}^3$ **onto** $\mathcal{S}_3$. The conclusion follows from (b) above.

7. The collection of all finite subsets of $\mathbb{N}$ is countable.

Pf: Extend items 1, 4, and 6 into an obvious induction proof that for each fixed $k \in \mathbb{N}$, the set $\mathbb{N}^k$ is countable and hence so is the family $\mathcal{S}_k$ consisting of all subsets of $\mathbb{N}$ with k or fewer elements. Now consider

$$\mathcal{F} = \{\emptyset\} \cup \bigcup_{k \in \mathbb{N}} \mathcal{S}_k.$$

The first set in this union is finite, and the second is countable (it's a countable union of countable sets). Recalling the “shift” idea from Hilbert’s Hotel, we conclude that $\mathcal{F}$ is countable. It’s clear that every element of $\mathcal{F}$ is a finite subset of $\mathbb{N}$, and that every finite subset of $\mathbb{N}$ is an element of $\mathcal{F}$.

Remark. Recall that every subset of a countable set is finite or countable. Thus item 5 above can be viewed as the most demanding version of the more flexible assertion that *every finite-or-countable union of finite-or-countable sets produces a finite-or-countable result.*

Definition. A set Y is **uncountable** if it is both infinite and not countable.

Remarks. • In our definitions, the word “uncountable” has a slightly different meaning than the phrase “not countable”. This is regrettable. Blame Rudin and move on.

- An effective way to prove that a given set $Y \neq \emptyset$ is uncountable is to show that every map $\phi: \mathbb{N} \rightarrow Y$ fails to be ONTO.
- If A is uncountable and $B \supseteq A$, then B is uncountable. (Remember: Every subset of a countable set is finite or countable.)

Theorem. *The set $\mathcal{P}(\mathbb{N})$ is uncountable.*

Proof. Match subsets of $\mathbb{N}$ with 0–1 sequences using a bijection b , writing $b(S) = (b_1, b_2, b_3, \dots)$ with $b_k = 1$ if $k \in S$ and $b_k = 0$ otherwise. This is clearly a bijection.

Given any $h: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$, let’s show that h is not onto. For convenience, work on $f = b \circ h$, which maps $\mathbb{N}$ to 0–1 sequences. Make a picture:

$$\begin{aligned} f(1) &= (b_1^1, b_2^1, b_3^1, b_4^1, \dots) \\ f(2) &= (b_1^2, b_2^2, b_3^2, b_4^2, \dots) \\ f(3) &= (b_1^3, b_2^3, b_3^3, b_4^3, \dots) \\ &\vdots \end{aligned}$$

Then define a sequence $y = (y_1, y_2, \dots)$ by flipping bits on the diagonal:

$$y_k = \begin{cases} 0, & \text{if } b_k^k = 1, \\ 1, & \text{if } b_k^k = 0, \end{cases} \quad k = 1, 2, 3, \dots$$

For each k , the bit-flip in position k guarantees that $y \neq f(k)$. Thus, $y \notin f(\mathbb{N})$, so f is not onto. Consequently the set $S = b^{-1}(y)$ is not in $h(\mathbb{N})$, so h is not onto. ///

2026-09-28

Corollary. (a) *The real interval $[0, 1]$ is uncountable.*

(b) *The set $\mathbb{R}$ is uncountable. So is every real interval $[a, b]$ with $b > a$.*

- (c) For every real interval $[a, b]$ with $b > a$, the subset consisting of irrational numbers is uncountable.

Proof. (Sketch) (a) For each number x in $[0, 1]$, form the binary expansion

$$x = 0.b_1b_2b_3\cdots = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad (b_k \in \{0, 1\} \ \forall k).$$

Some care is required here. Ambiguous cases arise when x has the form $p/2^k$ for some integer $p \in [1, 2^k - 1]$: in those cases, choose the representation that ends with an infinite string of 1's. Let A denote the set of binary sequences used to capture each $x \in [0, 1]$, and let C denote the set of binary sequences set aside by the choice made above. The set $A \cup C$ contains all 0–1 sequences, so it is uncountable. The set C corresponds to the collection of finite nonempty subsets of $\mathbb{N}$, which is known to be countable. The union of two countable sets is countable, so it must be that A is uncountable. But A is in one-to-one correspondence with $[0, 1]$, so the proof is complete.

- (b) Since $\mathbb{R}$ has an uncountable subset, namely $[0, 1]$, $\mathbb{R}$ itself must be uncountable too. For any real a and b with $a < b$, the function $f(t) = a + t(b - a)$ is a bijection from $[0, 1]$ to $[a, b]$. Since $[0, 1]$ is uncountable, so is $[a, b]$.
- (c) Clearly $[a, b] = ([a, b] \cap \mathbb{Q}) \cup ([a, b] \setminus \mathbb{Q})$. The set on the left is uncountable, so at least one of the sets on the right must be uncountable. But the first is a subset of the countable set $\mathbb{Q}$, so the second set must be uncountable. ////

Beyond Uncountability. Given any set X , consider the collection of all its subsets:

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U : U \subseteq X\}.$$

This is called “the power set” for X , and in some presentations it is denoted by 2^X .

Proposition. For any set X , every $f: X \rightarrow \mathcal{P}(X)$ is not onto.

Proof. Let any $f: X \rightarrow \mathcal{P}(X)$ be given. Recall Russell’s Paradox: thanks to ZFC, it’s legitimate to define a set $S \in \mathcal{P}(X)$ like this:

$$S = \{x \in X : x \notin f(x)\}.$$

Claim: $S \notin f(X)$.

Seeking a contradiction, suppose not, i.e., suppose $S \in f(X)$. This means there is an element $y \in X$ such that $S = f(y)$. Two cases arise:

- (i) $y \in S = f(y)$. Then, by def of S , $y \notin f(y)$ — contradiction.
- (ii) $y \notin S = f(y)$. Then, by def of S , $y \in S$ — contradiction.

Thus $S \notin f(X)$, so f is not onto. ////

Notation. Given sets X and Y , write either $|X| = |Y|$ or $X \sim Y$ when there exists a bijection from X to Y . It's straightforward to show that $\sim$ is an equivalence relation (reflexive, symmetric, and transitive); we refer to $|X|$ as “the cardinal number of X ”. Some important cardinal numbers have special names:

$$|\emptyset| = 0, \quad |\{1, 2, \dots, n\}| = n, \quad |\mathbb{N}| = \aleph_0, \quad |\mathbb{R}| = c.$$

The nonstrict inequality $|X| \leq |Y|$ means that there exists a one-to-one map $f: X \rightarrow Y$. We write $|X| < |Y|$ to mean both $|X| \leq |Y|$ and $|X| \neq |Y|$: this happens exactly when there is a one-to-one map $f: X \rightarrow Y$, but no such map is also onto.

With these symbols, we can restate “ X is finite or countable” in the symbolic form $|X| \leq \aleph_0 = |\mathbb{N}|$. Our findings above include the strict inequalities “ $c > \aleph_0$ ” and

$$0 < 1 < 2 < \dots < |\mathbb{N}| < |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < \dots$$

Remarks. For practice, prove that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$. The *continuum hypothesis* is the assertion that there is no set S for which $\aleph_0 < |S| < c$. This turns out to be undecidable in the ZFC axiom system: both the claim and its negation are compatible with ZFC! The details are beyond the scope of MATH 320.

Enrichment: The Schroeder-Bernstein Theorem

Material in this section will not be tested.

When dealing with inequalities between real numbers, the following statement feels “obvious” because it is so familiar; it's also a reasonably direct consequence of the defining properties of the symbols “ $\leq$ ” and “ $=$ ”:

$$\left[a \leq b \quad \text{and} \quad b \leq a \right] \implies a = b.$$

Now we have recycled the same symbols for use in comparing cardinal numbers, but given both of them new meanings in this context. So if two sets A and B are given, the following statement is similar to the previous one *in appearance only, and not in meaning*:

$$\left[|A| \leq |B| \quad \text{and} \quad |B| \leq |A| \right] \implies |A| = |B|. \quad (**)$$

Proving $(**)$ is by no means as simple as saying, “That's how inequalities work”! On the contrary, the only way to justify hijacking familiar comparison symbols like “ $\leq$ ” for new uses is to provide an independent proof that the new definitions work the way inequalities are expected to behave. That's the point of this section.

Proposition (Squeeze). *If both $A_0 \supseteq A_1 \supseteq A_2$ and $A_0 \sim A_2$, then $A_0 \sim A_1 \sim A_2$.*

Proof. The hypothesis allows for the extreme situations where either $A_0 = A_1$ or $A_1 = A_2$. In either case the conclusion is obvious, so we only need to handle the case where both inclusions in the setup are strict. In symbols, we will assume

$$A_0 \setminus A_1 \neq \emptyset \quad \text{and} \quad A_1 \setminus A_2 \neq \emptyset.$$

This setup is not essential, but it adds some reassuring nondegeneracy to the ingredients we build later.

Since $A_0 \sim A_2$ by hypothesis, there exists a bijection $f: A_0 \rightarrow A_2$. Use this f to define a sequence of nested sets:

$$A_3 = f(A_1), A_4 = f(A_2), \dots, A_{n+2} = f(A_n), \dots$$

Applying f to each set in the chain of inclusions $A_0 \supseteq A_1 \supseteq A_2$ gives $A_2 \supseteq A_3 \supseteq A_4$, and this process can be repeated inductively to reveal that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \dots$$

Since $A_0 \neq A_1 \neq A_2$ and f is 1-1, every inclusion in this infinite chain is strict.

Now observe that

$$A_0 = \left(\bigcup_{k=0}^{\infty} (A_k \setminus A_{k+1}) \right) \cup \left(\bigcap_{k=0}^{\infty} A_k \right). \quad (\dagger)$$

To prove $(\dagger)$, notice that the inclusion $\supseteq$ is obvious. For the reverse inclusion, $\subseteq$, pick any specific element x of A_0 . If this x lies in each set A_n , $n \in \mathbb{N}$, then x appears in the second parenthesized set on the right of $(\dagger)$. Otherwise, the set $\{n \in \mathbb{N} : x \notin A_n\}$ must be nonempty, so we can use the well-ordering property of $\mathbb{N}$ to define $k = (-1) + \min \{n : x \notin A_n\}$. For this k , we have $k \geq 0$ and $x \in A_k \setminus A_{k+1}$. Therefore x appears in the first parenthesized set on the right of $(\dagger)$.

With $(\dagger)$ in hand, let's simplify notation by writing

$$S = \bigcap_{k=0}^{\infty} A_k, \quad D_k = A_k \setminus A_{k+1}, \quad k \geq 0.$$

Now f is a bijection from A_k to A_{k+2} for every $k = 0, 1, 2, \dots$. Therefore, for every $k \geq 0$, we have

$$f(D_k) = f(A_k \setminus A_{k+1}) = f(A_k) \setminus f(A_{k+1}) = A_{k+2} \setminus A_{k+3} = D_{k+2}. \quad (\ddagger)$$

Then equation $(\dagger)$ expresses A_0 as a union of disjoint sets, as follows:

$$A_0 = \left(D_0 \cup D_1 \cup D_2 \cup \dots \right) \cup S.$$

Recall that $D_0 = A_0 \setminus A_1$, so

$$A_1 = \left(D_1 \cup D_2 \cup D_3 \cup \dots \right) \cup S.$$

To define a bijection $\phi: A_0 \rightarrow A_1$, we just make a ‘‘Hilbert’s Hotel’’-style shift of the even-subscripted sets in this disjoint union, leaving everything else unchanged:

$$\phi(x) = \begin{cases} f(x), & \text{if } x \in D_{2k} \text{ for some } k \geq 0, \\ x, & \text{otherwise.} \end{cases}$$

This ϕ is clearly 1-1 and onto, as required. ////

The following corollary is known as the **Schroeder-Bernstein Theorem (SBT)**.

Corollary (SBT). *Given sets A and B , suppose there are subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $A \sim B_1 \subseteq B$ and $B \sim A_1 \subseteq A$. Then $A \sim B$.*

Proof. Since $B \sim A_1$, there exists a bijection $f: B \rightarrow A_1$. Use it to define

$$A_2 \stackrel{\text{def}}{=} f(B_1) \subseteq f(B) = A_1 \subseteq A. \quad (*)$$

Then $A_2 \sim B_1$ thanks to f , while $B_1 \sim A$ is given, so $A_2 \sim A$. Using $(*)$ with the Proposition above, we have $A \sim A_1$. But $A_1 \sim B$ by hypothesis, so $A \sim B$. /////

Example. Let A denote the real interval $[0, 1)$, and let $B = [0, 1) \times [0, 1)$ denote the unit square in the Cartesian plane (with part of its boundary removed). Then $|B| = |A|$.

Proof. It suffices to define a 1–1 function $f: B \rightarrow A$. Why? Because then $A_1 \stackrel{\text{def}}{=} f(B)$ obeys $B \sim A_1 \subseteq A$, while $A \sim A \times \{0\} \subseteq B$ is obvious, so $A \sim B$ by Schroeder-Bernstein.

To define f , let any (x, y) in B be given. Write the decimal expansions

$$\begin{aligned} x &= 0.x_1 x_2 x_3 \dots = \sum_{k=1}^{\infty} \frac{x_k}{10^k}, & x_k &\in \{0, \dots, 9\}, \\ y &= 0.y_1 y_2 y_3 \dots = \sum_{k=1}^{\infty} \frac{y_k}{10^k}, & y_k &\in \{0, \dots, 9\}, \end{aligned}$$

always choosing the (unique) representation that does not end with an infinite string of 9's. Then let

$$f(x, y) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \dots = \sum_{k=1}^{\infty} \frac{x_k}{10^{2k-1}} + \sum_{k=1}^{\infty} \frac{y_k}{10^{2k}}.$$

The digit string specified for $f(x, y)$ will not end in an infinite chain of 9's, and $f(x, y) = f(x', y')$ clearly implies $(x, y) = (x', y')$, so f is a 1–1 mapping from B into A .

Q: Is this f surjective?

A: No. One element of $A = [0, 1)$ lying outside $f(B)$ is $1/11 = 0.\overline{09}$. /////