# IV. Order Completeness

UBC M320 Lecture Notes by Philip D. Loewen

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## A. Supremum and Infimum

Given any nonempty set $S \subseteq \mathbb{R}$, consider

$$A = \{a \in \mathbb{R} : \forall s \in S,\ a \le s\}.$$

If $A \neq \emptyset$, we say that the set $S$ is *bounded below*, and we call each $a \in A$ a *lower bound for $S$*. When $A = \emptyset$, we say that *$A$ has no lower bound*.

Symmetric terminology applies to

$$B = \{b \in \mathbb{R} : \forall s \in S,\ \ s \le b\}.$$

To say $S$ is *bounded above* means $B \neq \emptyset$; in this case, each $b \in B$ is an *upper bound* for $S$.

**Theorem (Order Completeness).** *In the context above,*

(a) *either $A = \emptyset$ or there exists $\alpha \in \mathbb{R}$ such that $A = (-\infty, \alpha]$, and*

(b) *either $B = \emptyset$ or there exists $\beta \in \mathbb{R}$ such that $B = [\beta, +\infty)$.*

*When $A \neq \emptyset$, we denote $\alpha = \inf(S)$; when $B \neq \emptyset$, we denote $\beta = \sup(S)$.*

*Proof.* (b) Up to a change in the letters involved, this property of $\mathbb{R}$ was the highlight of our construction.

(a) Similar, with supporting discussion later. ////

**Discussion/Notation.** The whole point of constructing $\mathbb{R}$ was to have confidence in the *existence* of the quantities $\alpha$ and $\beta$ above *as actual numbers*. We call $\alpha$ the *greatest lower bound* or *infimum* of $S$, and write $\alpha = \inf(S)$; likewise, $\beta$ is called the *least upper bound* or *supremum* of $S$, written $\beta = \sup(S)$. These are robust alternatives to the more fragile notions of "minimum" and "maximum" respectively.

**Supremum vs Maximum.** The generic notation $\max(S)$ refers to *the element $\widehat{s}$ of the set $S$* (if any) that obeys $\widehat{s} \ge s$ for all $s \in S$. For many sets $S \subseteq \mathbb{R}$, like $S = (0, 1)$, no such element exists. In the notation above, $S = (0, 1)$ gives $A = (-\infty, 0]$ and $B = [1, +\infty)$, so $\inf(S) = 0$ and $\sup(S) = 1$. Thus the infimum and supremum provide conceptually robust replacements for the more fragile ideas of minimum and maximum.

An equivalent and more often-seen characterization of $\beta = \sup(S)$ has 2 parts that give structure to typical proofs involving this concept:

(i) $\forall x \in S$, $x \le \beta$
(i.e., $\beta$ is *an upper bound* for $S$); and

(ii) $\forall \gamma < \beta$, $\exists x \in S :\ \ \gamma < x$
(i.e., every $\gamma$ less than $\beta$ is too small to be an upper bound for $S$—which makes $\beta$ *the least* upper bound).

**Extended Notation.** In the setup above, there are only 3 possibilities for the set of upper bounds for $S$, namely,

$$B = \{b \in \mathbb{R} : \forall s \in S,\ s \leq b\}.$$

- $B = \emptyset$ when the given set $S$ has no upper bound;
- $B = [\beta, +\infty)$ when the given set $S$ is nonempty and bounded above; and
- $B = (-\infty, +\infty)$ when $S = \emptyset$.

The middle scenario is the big one: it characterizes $\beta = \sup(S)$. A common-sense extension to cover the other cases would be to use the symbols $-\infty$ and $+\infty$ (**not real numbers!!**) as follows:

$\sup S = +\infty \Leftrightarrow$ the set $S$ has no upper bound,

$\sup S = -\infty \Leftrightarrow S = \emptyset$.

Of course symmetry then requires

$\inf S = +\infty \Leftrightarrow S = \emptyset$,

$\inf S = -\infty \Leftrightarrow$ the set $S$ has no lower bound,

This scheme gives a "value" (either numeric or symbolic) to the supremum and infimum for any set $S \subseteq \mathbb{R}$. Further, "$\inf S = -\sup(-S)$" holds in this extended interpretation. Let's check that the two-part characterization of $\beta = \sup(S)$ offered above is compatible with the extended interpretation.

- Case $\beta = -\infty$ arises only when $S = \emptyset$. In this case assertion (i) is valid because there are no elements $x$ in $S$ to falsify the strange-looking condition $x \leq -\infty$. And assertion (ii) is not wrong because there are no values $\gamma < -\infty$ for which one must find an element $x$ in $S$.
- Case $\beta = +\infty$ trivializes test (i), but reduces line (ii) to the assertion that for every $\gamma$ in $\mathbb{R}$, some element $x \in S$ has $x > \gamma$. Indeed this is equivalent to saying that $S$ has no upper bound.

This extension has some fine points: a set $S$ for which $\sup(S) = +\infty$ has no upper bound, so the phrase "least upper bound" no longer provides a perfectly accurate synonym for "supremum". (It's still a helpful mnemonic for structuring proofs, however.)

We also extend the definitions of $\sup(S)$ and $\inf(S)$ to cover sets $S$ in the extended real line $\mathbb{R} \cup \{\pm\infty\}$. (See Rudin 1.23, p. 11.) This adds some nuance to the simple situations above: now the statement that $\sup(S) = -\infty$ implies only that $S \subseteq \{-\infty\}$. (Given the extra information that $S \subseteq \mathbb{R}$, we would deduce $S = \emptyset$, as before.)

**Low-Hanging Fruit.**

- For any set $S \neq \emptyset$, $\inf(S) \leq \sup(S)$. (Indeed, any $s \in S$ obeys both $\inf(S) \leq s$ and $s \leq \sup(S)$. Chain these together.)

- $\inf(\emptyset) = +\infty$ and $\sup(\emptyset) = -\infty$ reverses the usual inequality most spectacularly.
- $\inf(S) = \sup(S)$ if and only if $S$ is a single-element set; that element equals both $\inf(S)$ and $\sup(S)$.
- Whenever $U, V \subseteq \mathbb{R}$,

$$U \subseteq V \implies \inf(V) \leq \inf(U) \text{ and } \sup(U) \leq \sup(V).$$

  This is obvious: $\inf(V)$ is a lower bound for $V$, so set inclusion makes it a lower bound for the subset $U$. And $\inf(U)$ is the *greatest* lower bound for $U$, so $\inf(U) \geq \inf(V)$. Etc. But note that the converse is most certainly false: let $U = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ and $V = \{-1, 1\}$ to see a situation where $\inf(V) < \inf(U)$ and $\sup(U) < \sup(V)$, but $U \not\subseteq V$.

## B. Lim Sup and Lim Inf

**Definition.** Let $(x_n)$ be a real sequence. We define the **limit superior** (or **upper limit**) and **limit inferior** (or **lower limit**) as

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} \left[\sup_{k\geq n} x_k\right], \qquad \liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \left[\inf_{k\geq n} x_k\right].$$

To expand the notation above, define for each $n$ the extended values

$$s_n = \sup\{x_k : k \geq n\} = \sup\{x_n, x_{n+1}, \dots\},$$
$$i_n = \inf\{x_k : k \geq n\} = \inf\{x_n, x_{n+1}, \dots\}.$$

The definitions mean

$$\limsup_{n\to\infty} x_n = \inf\{s_n : n \in \mathbb{N}\}, \qquad \liminf_{n\to\infty} x_n = \sup\{i_n : n \in \mathbb{N}\}.$$

Both quantities exist, with values in $\mathbb{R} \cup \{\pm\infty\}$, for absolutely any given sequence.

**Example.** Discuss in detail: (a) $x_n = 1/n$, (b) $x_n = (-1)^n + 1/n$, (c) $x_n = n$.

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Given any term-by-term inequality between two sequences, it's safe to "take the lim sup of both sides": the inequality will be preserved. That's the point of the next result.

**Lemma.** *For any sequences $(x_n)$ and $(y_n)$ in $\mathbb{R} \cup \{\pm\infty\}$ such that $x_n \leq y_n$ for all $n$, one has*

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n \quad \text{and} \quad \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n.$$

*Proof.* For each $n \in \mathbb{N}$, define

$$T_n(x) = \{x_n, x_{n+1}, x_{n+2}, \dots\},$$
$$i_n(x) = \inf T_n(x) = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\},$$
$$s_n(x) = \sup T_n(x) = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\},$$

with similar notation for $T_n(y)$, $i_n(y)$, $s_n(y)$. Observe that for each $n$, one has $i_n(x) \leq i_n(y)$. [Reason: Being *a* lower bound makes $i_n(x) \leq t$ for each $t \in T_n(x)$. The relationship between the given sequences makes $i_n(x) \leq t'$ for each $t' \in T_n(y)$. Among lower bounds for $T_n(y)$, the greatest is $i_n(y)$, so $i_n(x) \leq i_n(y)$.] Inserting the second inequality below gives

$$\forall n \in \mathbb{N},\ i_n(x) \leq i_n(y) \leq \sup_p i_p(y) = \sup_p \left[\inf_{k \geq p} y_k\right] = \liminf_{p\to\infty} y_p.$$

This shows that the quantity on the right dominates each element of the set $\{i_n(x) \,:\, n \in \mathbb{N}\}$. Since the supremum is the *least* upper bound,

$$\liminf_{n\to\infty} x_n = \sup_n i_n(x) \leq \liminf_{p\to\infty} y_p.$$

A very similar line of reasoning, starting with $s_n(x) \leq s_n(y)$ for each $n$, leads to the similar conclusion

$$\limsup_{n\to\infty} x_n = \inf_n s_n(x) \leq \inf_n s_n(y) = \limsup_{n\to\infty} y_n. \qquad ////$$

**Caution.** Limiting operations respect non-strict inequalities, but not strict ones. Look at $x_n = 1/n$. Clearly $x_n > 0$ for each $n$, yet $\limsup_n x_n = 0 = \liminf_n x_n$. When using this idea, one must to downgrade the inequality to non-strict.

The English phrases "lower limit" and "upper limit" are reasonable synonyms for "lim inf" and "lim sup", as shown next. Moreover, these complementary concepts completely capture our ordinary (and extended) concepts of limits.

**Proposition.** *Let $(x_n)$ be a sequence with values in $\mathbb{R} \cup \{\pm\infty\}$. Then*

(a) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$, *and*

(b) *for any $L \in \mathbb{R}\cup\{\pm\infty\}$, one has $x_n \to L$ if and only if* $\limsup_{n\to\infty} x_n = L = \liminf_{n\to\infty} x_n$.

*Proof.* For each $n \in \mathbb{N}$, define the tail set

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\},$$

and let

$$i_n = \inf T_n = \inf_{k\geq n} x_k, \quad s_n = \sup T_n = \sup_{k\geq n} x_k.$$

Observe that for each $n$,

$$i_n \leq s_n, \qquad i_n \leq i_{n+1}, \qquad s_n \geq s_{n+1}.$$

(a) For any pair $m, n \in \mathbb{N}$, pick any integer $N > \max\{m, n\}$ and combine the three observations above:

$$i_m \leq i_{m+1} \leq \cdots \leq i_N \leq s_N \leq \cdots \leq s_{n+1} \leq s_n.$$

This establishes the inequality $i_m \le s_n$ for all $m, n \in \mathbb{N}$. For fixed $n$, this shows that $s_n$ is *an* upper bound for the set $\{i_m : m \in \mathbb{N}\}$, so $s_n$ must dominate the set's *least* upper bound (which we recognize):

$$s_n \ge \sup_m i_m = \liminf_m x_m.$$

Since this holds for each $n$, the value on the right is a lower bound for the set $\{s_n : n \in \mathbb{N}\}$. Therefore it cannot exceed that set's *greatest* lower bound (which we recognize):

$$\liminf_m x_m \le \inf_n s_n = \limsup_n x_n.$$

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(b) ($\Rightarrow$) Suppose $x_n \to L$, with $L \in \mathbb{R} \cup \{+\infty\}$. For any real $\gamma < L$, the definition provides some $N \in \mathbb{N}$ such that

$$\gamma < x_n, \qquad \forall n > N.$$

That is, $\gamma$ is a lower bound for the set $T_{N+1}$, and this forces

$$\gamma \le i_{N+1} \le \sup_n i_n = \liminf_{n\to\infty} x_n.$$

Since this holds for all $\gamma < L$, we have $L \le \liminf_{n\to\infty} x_n$; in conjunction with part (a), we have

$$L \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n, \qquad \text{for } L \in \mathbb{R} \cup \{+\infty\}. \tag{$\dagger$}$$

Similarly, suppose $x_n \to L$ with $L \in \mathbb{R} \cup \{-\infty\}$. For any real $\lambda > L$, the definition provides some $N \in \mathbb{N}$ such that

$$x_n < \lambda, \qquad \forall n > N.$$

This implies that $\lambda$ is an upper bound for the set $T_{N+1}$, and therefore

$$\lambda \ge s_{N+1} \ge \inf_n s_n = \limsup_{n\to\infty} x_n.$$

Since this holds for all $\lambda > L$, we must have $L \ge \limsup_{n\to\infty} x_n$. Recalling part (a), we have

$$L \ge \limsup_{n\to\infty} x_n \ge \liminf_{n\to\infty} x_n, \qquad \text{for } L \in \mathbb{R} \cup \{-\infty\}. \tag{$\ddagger$}$$

Now consider the possibilities: if $L \in \mathbb{R}$, then both ($\dagger$) and ($\ddagger$) apply and the equations in (b) follow. If $L = -\infty$, then line ($\ddagger$) is enough to confirm (b), and if $L = +\infty$, then (b) follows from ($\dagger$).

($\Leftarrow$) Consider first the case where $L \in \mathbb{R} \cup \{+\infty\}$. Since $L = \sup_n i_n$, for any $\gamma < L$ there must exist some $N_1$ such that $i_{N_1} > \gamma$. In short

$$L > -\infty \implies \forall \gamma < L,\ \exists N_1 \in \mathbb{N} : \forall n \ge N_1, \quad x_n > \gamma. \tag{$*$}$$

On the other hand, suppose $L \in \mathbb{R} \cup \{-\infty\}$. Since $L = \inf_n s_n$, for any $\lambda > L$ there must exist some $N_2$ such that $s_{N_2} < \lambda$. Put succinctly,

$$L < +\infty \implies \forall \lambda > L,\ \exists N_2 \in \mathbb{N} : \forall n \geq N_2, \quad x_n < \lambda. \tag{$**$}$$

Now consider the possibilities: If $L = +\infty$, then line $(*)$ shows that $x_n \to +\infty$, and if $L = -\infty$ then line $(**)$ shows that $x_n \to -\infty$. If $L \in \mathbb{R}$, then both lines apply. Given any $\varepsilon > 0$, choose $\gamma = L - \varepsilon$ in $(*)$ to get an integer $N_1$ and choose $\lambda = L + \varepsilon$ in $(**)$ to get an integer $N_2$, and chain together the guaranteed inequalities:

$$\forall n > N \stackrel{\text{def}}{=} \max\{N_1, N_2\}, \qquad L - \varepsilon < x_n < L + \varepsilon.$$

This confirms the definition of $x_n \to L$. ////

## C. Cauchy Sequences

**Theorem.** *Every Cauchy sequence in $\mathbb{R}$ converges.*

*Proof.* Let $(x_n)$ be a Cauchy sequence with real values. As usual, define the tail sets

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+p}, \ldots\}, \qquad n \in \mathbb{N},$$

and let $i_n = \inf(T_n)$, $s_n = \sup(T_n)$.

Every Cauchy sequence is bounded, so there must exist some $M > 0$ such that

$$\forall n \in \mathbb{N}, \qquad -M \leq x_n \leq M.$$

Consequently

$$-M \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq M. \tag{$*$}$$

Let $\varepsilon > 0$ be given. Use the Cauchy property to produce some $N \in \mathbb{N}$ such that

$$\forall n \geq N,\ \forall p \in \mathbb{N}, \quad |x_{n+p} - x_n| < \varepsilon.$$

In particular, whenever $n \geq N$ we have $x_n - \varepsilon < x_{n+p} < x_n + \varepsilon$ for all $p \in \mathbb{N}$, so

$$x_n - \varepsilon \leq i_n \leq s_n \leq x_n + \varepsilon.$$

It follows that $s_n - i_n \leq (x_n + \varepsilon) - (x_n - \varepsilon) = 2\varepsilon$. Thus we have

$$\forall n \geq N, \quad s_n \leq 2\varepsilon + i_n.$$

Pick any specific $n \geq N$ and extend the above as usual:

$$\inf\{s_k : k \in \mathbb{N}\} \leq s_n \leq 2\varepsilon + i_n \leq 2\varepsilon + \sup\{i_k : k \in \mathbb{N}\}.$$

Recall the definitions: the inequality above implies

$$\limsup_{k\to\infty} x_k \leq 2\varepsilon + \liminf_{k\to\infty} x_k.$$

All these quantities are finite, and this works for arbitrary $\varepsilon > 0$, so indeed

$$\limsup_{k\to\infty} x_k \leq \liminf_{k\to\infty} x_k.$$

In conjunction with line $(*)$ above, we get $\limsup_{k\to\infty} x_k = \liminf_{k\to\infty} x_k$. ////

## D. Monotone Sequences

**Theorem (Monotone Convergence Property).** *For any real-valued sequence* $(x_n)$ *such that*

$$x_1 \le x_2 \le x_3 \le \cdots,$$

*exactly one of the following holds*:

(a) $x_n \to +\infty$ *as* $n \to \infty$;

(b) $x_n$ *converges.*

*Indeed,* $x_n \to \beta$ *as* $n \to \infty$*, where* $\beta = \sup\{x_k : k \in \mathbb{N}\}$.

**Terminology.** The condition in (a) characterizes a *nondecreasing* or *weakly increasing* sequence; a chain of strict inequalities would be required to qualify for the term *increasing.* Reversing all the inequalities defines the terms *nonincreasing*, *weakly decreasing*, or (when strict) *decreasing.* The term *monotone* or *monotonic* means *nonincreasing or nondecreasing.*

*Proof.* Let $\beta = \sup\{x_k : k \in \mathbb{N}\}$. Clearly $\beta \ge x_1 > -\infty$.

If $\beta = +\infty$, then conclusion (a) holds. [Indeed, pick any real $\gamma$. Then $\gamma < \beta$, so some sequence entry $x_N$ must exceed $\gamma$. Then by monotonicity, for all $n \ge N$ we have $x_n \ge x_N > \gamma$. This confirms the definition of $x_n \to +\infty$.]

If $\beta \in \mathbb{R}$, let $\varepsilon > 0$ be given. Then $\gamma = \beta - \varepsilon < \beta$, so the definition of sup gives some $N \in \mathbb{N}$ such that $x_N > \gamma = \beta - \varepsilon$. For every $n > N$, monotonicity gives

$$x_n \ge x_N > \beta - \varepsilon.$$

But of course we have $x_n \le \beta$ for each $n$, so that

$$\forall n > N, \quad \beta - \varepsilon < x_n \le \beta, \quad \text{i.e.,} \quad -\varepsilon < x_n - \beta \le 0 < \varepsilon. \qquad ////$$

**Corollary.** *For any real-valued sequence* $(x_n)$ *with*

$$x_1 \ge x_2 \ge x_3 \ge \cdots,$$

*exactly one of the following holds:*

(a) $x_n \to -\infty$ *as* $n \to \infty$;

(b) $x_n$ *converges.*

*Indeed,* $x_n \to \alpha$ *as* $n \to \infty$*, where* $\alpha = \inf\{x_k : k \in \mathbb{N}\}$.

*Proof.* Invent $y_n = -x_n$ and apply the previous theorem to the sequence $(y_n)$. ////

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**Discussion.** For any bounded real-valued sequence $(x_n)$ with the usual notation for the tail sets $T_n$ and extended values $i_n$, $s_n$, we always have

$$i_n \le i_{n+1}, \qquad i_n \le s_n, \qquad s_{n+1} \le s_n.$$

So the sequence of numbers $(i_n)$ is nondecreasing, and the result above shows that it approaches its supremum: in the extended-valued interpretation

$$\lim_{n\to\infty} i_n = \sup\{i_n : n \in \mathbb{N}\} = \liminf_{n\to\infty} x_n.$$

Similarly, the sequence $(s_n)$ is nonincreasing, and we have

$$\lim_{n\to\infty} s_n = \inf\{s_n : n \in \mathbb{N}\} = \limsup_{n\to\infty} x_n.$$

So another phrasing for "lim sup" could be "the limit of the tail sups". In particular, although the definitions make this non-obvious, the values of $\liminf x_n$ and $\limsup x_n$ are unaffected by changing any finite number of terms in the given sequence $(x_n)$.

**Example.** Some iterative process defines a sequence. Prove that it converges.

**Solution.** Use induction to prove that the sequence is monotonic and bounded, then apply the theorem.

**Example.** Recursively define $a_{n+1} = \sqrt{2a_n + 3}$, $a_1 = 4$. Prove that the sequence $(a_n)$ converges and find its limit.

**Solution.** Clearly all sequence elements are positive, and satisfy

$$a_{n+1}^2 = 2a_n + 3.$$

If the sequence converges to some number $\widehat{a}$, the limit laws imply that the sequences on both sides of this identity converge, with

$$\widehat{a}^2 = 2\widehat{a} + 3, \quad \text{i.e.,} \quad 0 = \widehat{a}^2 - 2\widehat{a} - 3 = (\widehat{a} - 3)(\widehat{a} + 1).$$

So either $\widehat{a} = -1$ or $\widehat{a} = 3$, but positivity of the sequence entries forces $\widehat{a} = 3$. Having a candidate for the limit we seek informs our approach. Since $a_1 = 4 > 3 = \widehat{a}$, we try to show

$$\forall n \in \mathbb{N}, \qquad 3 < a_{n+1} < a_n. \tag{$*$}$$

Prove this by induction.

Case $n = 1$: $a_2 = \sqrt{11}$ by substitution; clearly $a_2 < 4 = a_1$ and $a_2 > 3$.

Induction step: If $3 < a_{n+1} < a_n$ holds for some fixed $n$, consider

$$a_{n+2} = \sqrt{2a_{n+1} + 3}.$$

Using the known inequality gives

$$3 = \sqrt{2(3) + 3} < \sqrt{2a_{n+1} + 3} < \sqrt{2a_n + 3} = a_{n+1},$$

so we deduce $3 < a_{n+2} < a_{n+1}$.

By induction $(*)$ holds; by the theorem on monotone convergence, the sequence $(a_n)$ converges; by the calculation above, the limit value must be $\widehat{a} = 3$.

## E. Subsequences

These allow another description of lim sup and lim inf. Please read Rudin's presentation in items 3.5, 3.15–3.20.

**Definition.** (Subsequence) Let $x:\mathbb{N}\to X$ be a sequence with values in some set $X$. A sequence $y:\mathbb{N}\to X$ is a **subsequence of** $(x_n)$ if $y(k)=x(n(k))$ for some increasing function (sequence) $n:\mathbb{N}\to\mathbb{N}$, i.e.,

$$y_k = x_{n_k}, \quad \text{where} \quad 1 \le n_1 < n_2 < n_3 < \ldots.$$

**Alternative.** Every subsequence corresponds to an infinite subset $S$ of $\mathbb{N}$. It's possible to imagine notation that avoids putting subscripts on subscripts by subscripting the arrow in "$n\to\infty$" instead, so that we could write

$$x_{n_k} \to L \text{ as } k \to \infty \qquad \iff \qquad x_n \to L \text{ as } n \underset{S}{\to} \infty.$$

For $L\in\mathbb{R}$, the notation on the right would mean

$$\forall\varepsilon>0,\ \exists N\in\mathbb{N}:\ \forall n>N\,(n\in S),\ |x_n-L|<\varepsilon.$$

To get back to more standard notation, we would define indices $n_1, n_2, \ldots$ by enumerating $S$ in increasing order. This is a lovely idea, that captures the essence of subsequences, but let's stick closer to the notation standardized by Rudin for a little while longer.

**Example.** Subsequences of $x_n = n$ include

$$(1,3,5,7,\ldots),\ (2,4,6,8,\ldots),\ (2,3,5,8,13,21,\ldots),\ \text{even}\ (x_n)\ \text{itself},$$

but **not** $(3,1,4,1,5,9,\ldots)$ [order permuted and elements re-used].

**Proposition.** *Let $(x_n)_n$ be a sequence in $\mathbb{R}$. TFAE:*

(a) *$(x_n)$ converges.*

(b) *Every subsequence of $(x_n)$ converges.*

*Proof.* (a⇒b) Obvious.

(b⇒a) [Contraposition] If $(x_n)$ does not converge, let $n_k = k$: then the subsequence $(x_{n_k})_k = (x_k)_k$ also fails to converge. ////

**Proposition.** *Let $(x_n)$ be a real sequence; define $\mu = \liminf_{n\to\infty} x_n$, $M = \limsup_{n\to\infty} x_n$.*

(a) *If $\ell = \lim_k x_{n_k}$ for some subsequence $(x_{n_k})_k$, then $\mu \le \ell \le M$.*

(b) *There exist subsequences $\left(x_{n_j}\right)_j$ and $(x_{n_k})_k$ obeying $x_{n_j}\to\mu$ and $x_{n_k}\to M$.*

(*Note*: *values* $\pm\infty$ *allowed for* $\mu$, $\ell$, $M$ *throughout.*)

*Proof.* For each $n \in \mathbb{N}$, define

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}, \quad i_n = \inf T_n = \inf_{k\geq n} x_k, \quad s_n = \sup T_n = \sup_{k\geq n} x_k.$$

(a) For each $k \in \mathbb{N}$, $x_{n_k} \geq \inf\{x_j : j \geq n_k\} = i_{n_k}$. As $k \to \infty$, we have $x_{n_k} \to \ell$ and $i_{n_k} \to \mu$: hence $\ell \geq \mu$. Similarly, $x_{n_k} \leq s_{n_k}$ $\forall k$, so $\ell \leq M$.

(b) Let's build $(x_{n_k})_k$ with $x_{n_k} \xrightarrow[k]{} M$. (Similar arguments work for $\mu$.)

Case 1: $M = +\infty$. Here $\inf_n s_n = +\infty$, so $s_n = +\infty$ for all $n$. Pick $n_1 = 1$. Now $s_{n_1} = +\infty$ means that $T_{n_1} = \{x_k : k \geq 1\}$ has no upper bound. So pick $n_2 \geq n_1$ such that $x_{n_2} > 2$. Take care to choose $n_2 > n_1$. After choosing $n_1 < \ldots < n_k$, note that $s_{n_k} = +\infty$, so there must exist some $n_{k+1} > n_k$ such that $x_{n_{k+1}} > k+1$. By induction, this defines a whole subsequence along which $x_{n_k} > k$ for each $k \in \mathbb{N}$. Clearly $x_{n_k} \to +\infty$ as $k \to \infty$.

Case 2: $M \in \mathbb{R}$. Pick $n_1 = 1$. For each $k \geq 2$, define $r_k = M - 1/k$ and $R_k = M + 1/k$ and apply this two-step reasoning:

(i) Since $R_k > M = \inf_n s_n$, some $N_k$ must obey $R_k > s_{N_k}$. In particular, $R_k > x_n$ for all $n \geq N_k$. Define $\widehat{N}_k = 1 + \max\{N_k, n_1, \dots, n_{k-1}\}$.

(ii) Since $r_k < M = \inf_n s_n$, we certainly have $r_k < s_{1+\widehat{N}_k}$. Hence there exists $n_k > \widehat{N}_k$ such that $x_{n_k} > r_k$. Since $n_k > N_k$ also, it is eligible for the inequality in (i). Thus

$$M - \frac{1}{k} = r_k < x_{n_k} < R_k = M + \frac{1}{k}.$$

This construction works for all $k$: it guarantees both $n_k < n_{k+1}$ for all $k$ and (by the Squeeze Theorem) $x_{n_k} \to M$ as $k \to \infty$.

Case 3: $M = -\infty$. Exercise. [We always have $\mu \leq M$. So $M = -\infty$ forces $\mu = -\infty$, i.e., $M = \mu$. This is equivalent to $x_n \to -\infty$ by some earlier result. The original sequence is a subsequence of itself.] ////

## F. Proof Tips

Many homework problems request a proof that $x_n \to L$ for some given sequence $(x_n)$ and extended real value $L$. This amounts to proving the inequalities

$$L \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq L. \tag{$*$}$$

(The middle inequality always holds; the outer ones might be "interesting".)

Let's discuss the case of $(*)$ in which $L$ is a real number. One way to prove it starts by wrapping the number $L$ in an arbitrary open interval $(\alpha, \beta)$. That is, we invent symbols $\alpha$ and $\beta$ for which $\alpha < L < \beta$ is all we know. We treat these new elements as constants at first. That gives us access to positive numbers $\varepsilon_1 = L - \alpha$ and $\varepsilon_2 = \beta - L$ that can provide the margins or tolerances that are so typical in our definitions. We put these to work, trying to establish the following relaxed version of the chain of inequalities in $(*)$:

$$\alpha \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le \beta. \tag{$\dagger$}$$

If we can get $(\dagger)$ for arbitrary choices of $\alpha$ and $\beta$ as above, then we can easily harvest $(*)$. Just imagine inventing sequences $\alpha_k = L - 1/k$ and $\beta_k + 1/k$ and using those in $(\dagger)$ to get

$$\forall k \in \mathbb{N}, \qquad L - \frac{1}{k} \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le L + \frac{1}{k}.$$

In the limit as $k \to \infty$, the numbers sandwiched in the middle don't depend on $k$, and the other (non-strict) inequalities are respected by the limiting operation, so $(*)$ is the result. (This is essentially the Squeeze Theorem discussed previously.)

Having seen this idea once, it gets tedious to repeat the story about the sequences $\alpha_k$ and $\beta_k$ every single time. We can just show the inequalities in $(\dagger)$ and say, "Since this holds for arbitrary $\alpha < L$ and $\beta > L$, we can now let $\alpha \to L^-$ and $\beta \to L^+$ to obtain $(*)$."

The approach above admits natural adaptations to handle proofs in which the desired values are $L = -\infty$ or $L = +\infty$, or in which there is only one inequality to prove.