

V. Vectors

UBC M320 Lecture Notes by Philip D. Loewen

For any $d \in \mathbb{N}$, we write $\mathbb{R}^d$ for the set of ordered d -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_d), \quad x_j \in \mathbb{R}.$$

(We don't write the parentheses when $d = 1$.) There are d standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_d$: In $\mathbf{e}_p$, all the components are 0 except for a 1 in position p . The standard operations on $\mathbb{R}^d$ are vector addition

$$(x_1, x_2, \dots, x_d) + (y_1, y_2, \dots, y_d) = (x_1 + y_1, x_2 + y_2, \dots, x_d + y_d),$$

multiplication by real scalars

$$c(x_1, x_2, \dots, x_d) = (cx_1, cx_2, \dots, cx_d),$$

and the dot product:

$$(x_1, x_2, \dots, x_d) \bullet (y_1, y_2, \dots, y_d) = x_1 y_1 + x_2 y_2 + \dots + x_d y_d = \sum_{j=1}^d x_j y_j.$$

The dot product induces the “norm”

$$|\mathbf{x}| = \sqrt{\mathbf{x} \bullet \mathbf{x}} = \sqrt{\sum_{j=1}^d x_j^2},$$

which makes $\mathbb{R}^d$ into **Euclidean d -space**. (This definition is fully compatible with the usual absolute value on $\mathbb{R} = \mathbb{R}^1$, since $|x| = \sqrt{x^2}$ holds for each real number x .) We have $|\mathbf{e}_p| = 1$ for each $p \in \{1, \dots, d\}$. Fundamentals in $\mathbb{R}^d$ are listed in Rudin's Theorem 1.37: highlights, valid for arbitrary $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^d$, include

(i) $|\mathbf{x}|^2 = \mathbf{x} \bullet \mathbf{x}$ (often useful in proofs),

(ii) the Schwarz inequality

$$|\mathbf{x} \bullet \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|, \quad \text{with “=” iff } \alpha \mathbf{x} + \beta \mathbf{y} = \mathbf{0} \text{ for some real } \alpha, \beta \text{ not both zero,}$$

(iii) the triangle inequalities

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|, \quad |\mathbf{x} - \mathbf{y}| \geq \left| |\mathbf{x}| - |\mathbf{y}| \right|.$$

Discussion. (i) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$\begin{aligned} 0 \leq |\mathbf{x} - t\mathbf{y}|^2 &= (\mathbf{x} - t\mathbf{y}) \bullet (\mathbf{x} - t\mathbf{y}) \\ &= \mathbf{x} \bullet \mathbf{x} - t\mathbf{y} \bullet \mathbf{x} - t\mathbf{x} \bullet \mathbf{y} + t^2 \mathbf{y} \bullet \mathbf{y} \\ &= |\mathbf{x}|^2 - 2t\mathbf{x} \bullet \mathbf{y} + t^2 |\mathbf{y}|^2. \end{aligned}$$

If $\mathbf{y} \neq \mathbf{0}$, the right-hand side is a quadratic polynomial in t with at most one real root. Hence, its discriminant cannot be positive:

$$\begin{aligned} (-2\mathbf{x} \bullet \mathbf{y})^2 - 4(|\mathbf{x}|^2)(|\mathbf{y}|^2) &\leq 0, \\ |\mathbf{x} \bullet \mathbf{y}|^2 &\leq (|\mathbf{x}| |\mathbf{y}|)^2. \end{aligned}$$

Taking square roots of both sides gives the Schwarz inequality. (If $\mathbf{y} = \mathbf{0}$, the inequality is obvious.) In this argument, the discriminant equals 0 if and only if the given quadratic polynomial has a repeated real root: this happens exactly when $\mathbf{x} = t\mathbf{y}$ for some real t ; the summary statement above expresses this in a symmetric style that correctly captures the situations where $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.

(ii) In the calculation above, we may extract

$$|\mathbf{x} - t\mathbf{y}|^2 = |\mathbf{x}|^2 - 2t\mathbf{x} \bullet \mathbf{y} + t^2|\mathbf{y}|^2, \quad \text{i.e.,} \quad |\mathbf{x} - t\mathbf{y}|^2 - t^2|\mathbf{y}|^2 = \mathbf{x} \bullet \mathbf{x} - 2t\mathbf{x} \bullet \mathbf{y}.$$

Substituting $c = -2t$ gives an alternative presentation for the expression on the left below:

$$\mathbf{x} \bullet \mathbf{x} + c\mathbf{y} \bullet \mathbf{x} = \left| \mathbf{x} + \left(\frac{c}{2}\right)\mathbf{y} \right|^2 - \left(\frac{c}{2}\right)^2 |\mathbf{y}|^2.$$

This extends the algebraic idea of “completing the square” to scalar-valued quadratic functions with a vector variable $\mathbf{x}$.

(iii) The triangle inequality follows from the Schwarz inequality: for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \bullet \mathbf{y} + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}| |\mathbf{y}| + |\mathbf{y}|^2 \quad (\text{by Schwarz, since } p \leq |p| \ \forall p \in \mathbb{R}) \\ &= \left| |\mathbf{x}| + |\mathbf{y}| \right|^2. \end{aligned}$$

Taking square roots of both sides gives the desired result.

For arbitrary $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, the result above implies

$$|\mathbf{v}| = |\mathbf{u} + (\mathbf{v} - \mathbf{u})| \leq |\mathbf{u}| + |\mathbf{v} - \mathbf{u}|.$$

Rearrangement shows

$$|\mathbf{v}| - |\mathbf{u}| \leq |\mathbf{v} - \mathbf{u}|.$$

The same argument works with the variables $\mathbf{u}$ and $\mathbf{v}$ swapped: this leaves the right side unchanged, and switches the sign on the left. Thus we have

$$\left| |\mathbf{v}| - |\mathbf{u}| \right| \leq |\mathbf{v} - \mathbf{u}|. \quad \text{////}$$