VI. Hausdorff Topological Spaces

A. Open Sets – Three Steps

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Step One: Euclidean k-space. Recall that $\mathbb{R}^k = \{x = (x_1, \ldots, x_k) : x_i \in \mathbb{R}\}$ is the set of k-element tuples of real numbers. (When $k = 1$, we ignore the difference in types between a 1-element tuple (x) and the ordinary real number x inside it, and casually pretend $\mathbb{R}^1 = \mathbb{R}$.)

Definition. Given $x \in \mathbb{R}^k$ and $r > 0$, the **open ball of centre x and radius r** is

$$\mathbb{B}[x; r) = \left\{ y \in \mathbb{R}^k \,:\, |y - x| < r \right\}.$$

A subset U of $\mathbb{R}^k$ is **open** if and only if

$$\forall x \in U,\ \exists r > 0 \ \ :\ \ \mathbb{B}[x; r) \subseteq U. \qquad (*)$$

(Intuition: Each point x in U comes with some radius r, maybe small but guaranteed to be *positive*, such that one can move away from x in any direction at all without leaving the set U ... provided the length of the excursion is not more than r.)

Remark. For fixed $\overline{x}$ in $\mathbb{R}^k$ and $\overline{r} > 0$, the set $U = \mathbb{B}[\overline{x}; \overline{r})$ is an "open set" (so the name "open ball" is well-deserved).

> Indeed, pick any x in U. Then $|x - \overline{x}| < \overline{r}$, so $\varepsilon \stackrel{\text{def}}{=} \overline{r} - |x - \overline{x}| > 0$. To show that $\mathbb{B}[x; \varepsilon) \subseteq U$, pick any $z \in \mathbb{B}[x; \varepsilon)$: then $|z - x| < \varepsilon$, so
>
> $$|z - \overline{x}| = |z - x + x - \overline{x}| \leq |z - x| + |x - \overline{x}| < \overline{r}.$$
>
> That is, $z \in \mathbb{B}[\overline{x}; \overline{r}) = U$. Since z is arbitrary in $\mathbb{B}[x; \varepsilon)$, this reasoning shows that $\mathbb{B}[x; \varepsilon) \subseteq U$, as required.

Remark. In Euclidean 2-space, $U = \{(x, y) : y > 0\}$ is open; $C = \{(x, y) : y \geq 0\}$ is not open but $C^c \stackrel{\text{def}}{=} \{(x, y) : y < 0\}$ is open; $L = \{(x, y) : y > 0 \text{ or } x \geq 0\}$ is not open and L^c is not open either.

Notation. 1. The collection of sets $\mathcal{T} = \left\{ U \subseteq \mathbb{R}^k \,:\, U \text{ is an open set} \right\}$ defines the usual "topology" on $\mathbb{R}^k$.

2. For a given point $x \in \mathbb{R}^k$, the collection of sets

$$\mathcal{N}(x) = \{S \in \mathcal{P}(X) \,:\, \text{ some } U \in \mathcal{T} \text{ obeys } x \in U \subseteq S\}$$

is the family of "neighbourhoods of x". [Some writers insist on using only open sets as "neighbourhoods". There is no good reason for this restriction.]

Convergence. The statement "$x_n \to \widehat{x}$ in $\mathbb{R}^k$" can be expressed equivalently as

(a) $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ \forall n > N,\ x_n \in \mathbb{B}[\widehat{x}; \varepsilon)$.

(b) $\forall S \in \mathcal{N}(\widehat{x}),\ \exists N \in \mathbb{N}\ :\ \forall n > N,\ x_n \in S$.

(b⇒a): Assume (b). Given $\varepsilon > 0$, take $S = \mathbb{B}[\widehat{x}; \varepsilon)$: then $S \in \mathcal{N}(x)$, so conclusion of (a) follows.

(a⇒b): Assume (a). Given $S \in \mathcal{N}(\widehat{x})$, choose $\varepsilon > 0$ such that $\mathbb{B}[\widehat{x}; \varepsilon) \subseteq S$. Apply (a) to this ε. Get back N such that all $n > N$ give $x_n \in \mathbb{B}[\widehat{x}; \varepsilon)$. For these same n, we have $x_n \in S$, by choice of ε above.

Properties of Open Sets. The numbering here anticipates properties (HTS1)–(HTS4) below.

(i) Both $\emptyset$ and $\mathbb{R}^k$ are open. In symbols,

$$\emptyset \in \mathcal{T} \quad \text{and} \quad \mathbb{R}^k \in \mathcal{T}.$$

(ii) Any union of open sets is open. That is, if $\mathcal{G}$ is any collection of open sets, then defining $U = \bigcup \mathcal{G}$ produces an open set. Recall that $\bigcup \mathcal{G} = \bigcup_{G \in \mathcal{G}} G$. In symbols,

$$\mathcal{G} \subseteq \mathcal{T} \implies \bigcup \mathcal{G} \in \mathcal{T}.$$

(iii) Any intersection of **finitely many** open sets is open. In symbols,

$$\Big[n \in \mathbb{N}, \quad U_1, \ldots, U_n \in \mathcal{T} \Big] \implies U_1 \cap \cdots \cap U_n \in \mathcal{T}.$$

(iv) Distinct points can be given disjoint neighbourhoods. In symbols,

$$\forall x, y \in \mathbb{R}^k, \qquad x \neq y \implies \Big[\exists U, V \in \mathcal{T} \,:\, x \in U,\ y \in V,\ U \cap V = \emptyset \Big].$$

[Sketch justifications; note that finiteness is essential in (iii), because $\bigcap_n (-\infty, 1/n) = (-\infty, 0]$ is not open.]

(i) Clearly $X = \mathbb{R}^k$ is open; and $U = \emptyset \in \mathcal{T}$ because there are no points x in U to falsify $(*)$.

(ii) Let $\mathcal{G}$ be any subset of $\mathcal{T}$, and consider $U = \bigcup \mathcal{G} = \bigcup_{G \in \mathcal{G}} G$. If $x \in U$, then there exists $G \in \mathcal{G}$ such that $x \in G \in \mathcal{T}$. By $(*)$ for G, $\exists \varepsilon > 0$ s.t. $\mathbb{B}[x; \varepsilon) \subseteq G \subseteq U$.

(iii) Use induction, defining

$$P(n): \qquad \mathcal{T} \text{ contains any intersection of } n \text{ members of } \mathcal{T}.$$

Start with $n = 2$. If $U_1, U_2 \in \mathcal{T}$, let $U = U_1 \cap U_2$. For any x in U, we have

$$x \in U_1 \implies \exists \varepsilon_1 > 0 : \mathbb{B}[x; \varepsilon_1) \subseteq U_1,$$
$$x \in U_2 \implies \exists \varepsilon_2 > 0 : \mathbb{B}[x; \varepsilon_2) \subseteq U_2.$$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ to see $\mathbb{B}[x; \varepsilon) \subseteq U_1 \cap U_2 = U$, as required.
Now assume $P(n)$ is true, and let $U_1, \ldots, U_{n+1}$ be given members of $\mathcal{T}$. Then

$$U \stackrel{\text{def}}{=} \bigcap_{k=1}^{n+1} U_k = [U_1 \cap U_2 \cap \cdots U_n] \cap U_{n+1}$$

displays U as an intersection of two sets: the first is open by $P(n)$, and the second (U_{n+1}) is open by hypothesis. Hence U is open by $P(2)$. This establishes $P(n+1)$, and it follows from the principle of mathematical induction that $P(k)$ is true for all $k \in \mathbb{N}$.

(iv) If $x, y \in \mathbb{R}^k$ obey $x \neq y$, then $|y - x| > 0$. So let $\varepsilon = |y - x|/2$: then

$$\mathbb{B}[x; \varepsilon) \cap \mathbb{B}[y; \varepsilon) = \emptyset.$$

The triangle inequality shows this. Indeed, any specific $z \in \mathbb{B}[x; \varepsilon)$ obeys $|z - x| < \varepsilon$, so

$$|z - y| \geq |y - x| - |z - x| > 2\varepsilon - \varepsilon = \varepsilon.$$

That is, $z \notin \mathbb{B}[y; \varepsilon)$. Remember that both $\mathbb{B}[x; \varepsilon)$ and $\mathbb{B}[y; \varepsilon)$ are open. ////

Step 2: Metric Spaces.

Definition. Given a nonempty set X, a function $d\colon X \times X \to \mathbb{R}$ is called a **metric** if

(a) $d(x, y) \geq 0$ $\forall x, y \in X$, with equality iff $x = y$ [positivity];

(b) $d(x, y) = d(y, x)$ $\forall x, y \in X$ [symmetry];

(c) $d(x, z) \leq d(x, y) + d(y, z)$ $\forall x, y, z \in X$ [triangle inequality].

In this case the pair (X, d) is called a **metric space**.

Examples. • For any nonempty set X (including $\mathbb{R}^k$!),

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

defines **the discrete metric**.

- For $X = \mathbb{R}^k$, $d(x, y) = |y - x|$ defines **the Euclidean metric**. Properties (a)–(b) are clear; we proved (c) earlier this term. Unless otherwise specified, this is the metric we mean whenever we write just "$\mathbb{R}^k$".

- For $X = \mathbb{R}^k$, other famous metrics include

$$\begin{aligned} d_1(x,y) &= \sum_{n=1}^{k} |x_n - y_n|, \\ d_\infty(x,y) &= \max\left\{|x_n - y_n| \,:\, n = 1, 2, \ldots, k\right\}, \\ d_p(x,y) &= \left(\sum_{n=1}^{k} |x_n - y_n|^p\right)^{1/p} \qquad p \geq 1. \end{aligned}$$

  Properties (a)–(c) are obvious for the first two above; for general $p > 1$, the triangle inequality for d_p is known as Minkowski's Inequality. (The fact that it is named after somebody suggests, correctly, that it is not obvious.)

- For $X = \ell^2$, the set of sequences $x = (x_1, x_2, \ldots)$ obeying $\displaystyle\sum_{n=1}^{\infty} |x_n|^2 < +\infty$, declaring

$$d(x,y) = \sqrt{\sum_{n=1}^{\infty} |y_n - x_n|^2}$$

  defines a metric that shares many features with the Euclidean metric on $\mathbb{R}^k$. Various supporting facts are explored on HW.

- For the set X of bounded functions $x\colon [0,1] \to \mathbb{R}$,

$$d(x,y) = \sup\left\{|y(t) - x(t)| \,:\, t \in [0,1]\right\}$$

  defines a metric (home practice!).

Notation. Taken together, the set X and metric d provide a complete context for some problems. So we call the pair (X, d) a "metric space". In this context, we define various "balls" for any $x \in X$ and $r > 0$:

$$\begin{aligned} \mathbb{B}[x;r) &= \{y \in X \,:\, 0 \leq d(x,y) < r\}, \\ \mathbb{B}[x;r] &= \{y \in X \,:\, 0 \leq d(x,y) \leq r\}, \\ \mathbb{B}(x;r) &= \{y \in X \,:\, 0 < d(x,y) < r\}, \\ \mathbb{B}(x;r] &= \{y \in X \,:\, 0 < d(x,y) < r\}. \end{aligned}$$

The inequalities requiring $d(x,y) \geq 0$ in the first two lines are redundant. Showing them here is supposed to help explain the interpretation of various parenthesis/bracket shapes.

The Open Sets. In a metric space (X, d), let $\mathcal{T}$ denote the collection of all subsets $G \subseteq X$ with this property:

$$\forall x \in G,\ \exists r > 0 \,:\, \mathbb{B}[x;r) \subseteq G.$$

A set G is called *open* if and only if $G \in \mathcal{T}$.

Proposition. *In any metric space (X, d), properties* (i)–(iv) *enumerated earlier for open sets in $\mathbb{R}^k$ remain valid.*

Proof. Just reread the sketchy justifications above, changing $|y-x|$ to $d(x, y)$ wherever the former appears. ////

Convergence. In a metric space (X, d), a sequence (x_n) and an element $\widehat{x}$ are related by saying $\widehat{x} = \lim_{n\to\infty} x_n$ (or $x_n \to \widehat{x}$ as $n \to \infty$) exactly when

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} : \forall n > N,\ d(x_n, \widehat{x}) < \varepsilon.$$

This is perfectly analogous with the familiar situation in $\mathbb{R}^k$. A given sequence (x_n) is called *convergent* if and only if there exists $\widehat{x} \in X$ such that $x_n \to \widehat{x}$ as $n \to \infty$.

Note that the limiting statement $x_n \to \widehat{x}$ in (X, d) is logically equivalent to the limiting statement $d(x_n, \widehat{x}) \to 0$ in $\mathbb{R}$. So any method for showing that a nonnegative sequence of real numbers converges to 0 may be useful in the study of metric spaces. (The Squeeze Theorem is a leading candidate for this.)

In the presence of a metric, most topological concepts admit simple (and useful!) characterizations built from sequences. Here is a first result of this type.

Proposition. *In a metric space (X, d), with subset A, the following are equivalent:*

(a) *A is an open set.*

(b) *For every $x \in A$, and every sequence (x_n) obeying $x_n \to x$, one has $x_n \in A$ for all n sufficiently large. That is,*

$$\exists N \in \mathbb{N} : \forall n > N,\ x_n \in A.$$

(*Constant sequences are allowed.*)

Proof. (a⇒b) Suppose A is open. Pick any $x \in A$ and any sequence (x_n) converging to x. Since A is open and $x \in A$, there exists $\varepsilon > 0$ so small that $\mathbb{B}[x; \varepsilon) \subseteq A$. Use this $\varepsilon > 0$ as the tolerance in the convergence definition: this gives some $N \in \mathbb{N}$ such that for all $n > N$, one has $d(x_n, x) < \varepsilon$. This inequality means precisely that $x_n \in \mathbb{B}[x; \varepsilon) \subseteq A$, as required.

(b⇒a) Contraposition: Suppose A is not open. This means that there exists some point $x \in A$ such that

$$\begin{aligned} &\forall \varepsilon > 0,\ \mathbb{B}[x; \varepsilon) \not\subseteq A, \\ \text{i.e.,}\quad &\forall \varepsilon > 0,\ \mathbb{B}[x; \varepsilon) \cap A^c \neq \emptyset. \end{aligned}$$

For each $n \in \mathbb{N}$, use this statement with $\varepsilon = 1/n$ to see that $\mathbb{B}[x; 1/n)$ contains a point $x_n \notin A$. This produces a sequence x_n for which $x_n \notin A$ for all n and yet $d(x_n, x) < 1/n$ so that $x_n \to x$. We have shown "$\neg$(a) $\implies$ $\neg$(b)", which is logically equivalent to "(a) $\implies$ (b)". ////

Step Three: Hausdorff Topological Spaces.

Definition. A **topological space** has two ingredients: a set X, and a family $\mathcal{T}$ of subsets of X called "the open sets", related by properties (HTS1)–(HTS3) below:

(HTS1) Both $\emptyset$ and X are open (i.e., $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$).

(HTS2) Any union of open sets is open. That is, for any subset $\mathcal{G}$ of $\mathcal{T}$, one has $\bigcup \mathcal{G} \in \mathcal{T}$.

(HTS3) Any intersection of **finitely many** open sets is open. That is, if $N \in \mathbb{N}$ and $U_1, \ldots, U_N \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_N \in \mathcal{T}$.

We will deal only with **Hausdorff** topological spaces, which also obey

(HTS4) Whenever x and y are distinct points of X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

We write HTS instead of Hausdorff Topological Space.

We have already verified conditions (HTS1)–(HTS4) for the topology generated by any metric space. These notes depart slightly from the textbook presentation by avoiding explicit references to metrics wherever possible. This perspective is useful because it takes almost no extra work, and produces results that can be used in HTS's that involve topologies that do not originate with a metric. (Such HTS's actually exist and have uses in the wider world!)

IMPORTANT
All HTS definitions and theorems are available in any metric topology.

Example. (a) Discrete Topology. For any X one may consider

$$\mathcal{T} = \{U \,:\, U \subseteq X\}\,.$$

In this topology, every set is open, and conditions (i)–(iv) are obvious.

(b) A non-Hausdorff topology (the only one we'll ever consider!). Take $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1\}, X\}$. Conditions (i)–(iii) clearly hold, but (iv) fails on $x = 2$, $y = 3$.

(c) The usual topology on $\mathbb{R}^k$. Take $X = \mathbb{R}^k$. Define $\mathcal{T}$ by saying that $U \in \mathcal{T}$ iff every point x in U has this property: for some $\varepsilon > 0$ (depending on x),

$$U \supseteq \mathbb{B}[x; \varepsilon) \stackrel{\text{def}}{=} \left\{y \in \mathbb{R}^k \,:\, |y - x| < \varepsilon\right\}. \qquad (*)$$

Conditions (i)–(iv) are demonstrated in Section A.

B. Neighbourhoods and Interior

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Definition. Let $(X, \mathcal{T})$ be a HTS. For each $x \in X$, let $\mathcal{N}(x)$ denote *the set of all neighbourhoods of x,* defined by

$$S \in \mathcal{N}(x) \iff \exists U \in \mathcal{T} \,:\, x \in U \subseteq S.$$

(Every open set containing x belongs to $\mathcal{N}(x)$. Often some non-open sets do, too.)

Lemma. *In a HTS $(X, \mathcal{T})$, with $A \subseteq X$, TFAE*:

(i) *A is open;*

(ii) *For each $x \in A$, one has $A \in \mathcal{N}(x)$.*

Proof. (i⇒ii) For any x in A, we obviously have $x \in A \subseteq A$. When A is open, we have $A \in \mathcal{T}$, so the definition of $A \in \mathcal{N}(x)$ stated above holds with $U = A$.

(ii⇒i) For each x in A, choose an open set U_x obeying both $x \in U_x$ and $U_x \subseteq A$. Then let

$$U = \bigcup_{x \in A} U_x.$$

As a union of open sets, U is open; since $U_x \subseteq A$ for all x, we have $U \subseteq A$; and for every x in A, we have $x \in U_x \subseteq U$, so $A \subseteq U$. Thus $A = U$, and A is open. ////

Definition. Let A be any set in a HTS $(X, \mathcal{T})$. The **interior of** A is the set

$$A^\circ = \{x \in A : x \in U \text{ and } U \subseteq A \text{ for some } U \in \mathcal{T}\}.$$

(Alternative notation: $A^\circ = \operatorname{int} A = \{x \in A : A \in \mathcal{N}(x)\}$.)

Clearly, whenever $A \subseteq B$ then $A^\circ \subseteq B^\circ$. Intuitively, A° is the largest open subset of A, in the sense captured by parts (a)–(b) of the following result.

Proposition. (a) *A° is open, and $A^\circ \subseteq A$.*

(b) *If G is open and $G \subseteq A$, then $G \subseteq A^\circ$.*

(c) *A is open if and only if $A = A^\circ$.*

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Proof. (a) Suppose $z \in A^\circ$. Then $\exists U \in \mathcal{N}(z) \cap \mathcal{T}$ such that $U \subseteq A$. Every x in U obeys

$$U \in \mathcal{N}(x) \text{ and } U \subseteq A, \qquad \text{i.e.,} \qquad x \in A^\circ.$$

Hence $U \subseteq A^\circ$. Since z is arbitrary in A°, the Lemma above implies A° is open.

(b) For any open $G \subseteq A$, pick any $z \in G$. Then $G \in \mathcal{N}(z)$ and $G \subseteq A$, so $z \in A^\circ$. Hence $G \subseteq A^\circ$.

(c) (⇐) Obvious from (a). (⇒) $A \supseteq A^\circ$ holds for any A. When A is open, choosing $G = A$ in part (b) gives the reverse inclusion. Hence equality holds. ////

Example. If $a < b$ in $\mathbb{R}$, then $[a, b]^\circ = [a, b)^\circ = (a, b]^\circ = (a, b)^\circ = (a, b)$; $\mathbb{Q}^\circ = \emptyset$.

Remark. A° is the largest open subset of A. The smallest open subset is $\emptyset$; the largest open superset is X. The smallest open superset may fail to exist. For example, any open superset U of $[a, b]$ must contain an open interval of the form $(a - 1/n, b + 1/n)$

for some $n \in \mathbb{N}$; boosting n to $n+1$ defines a new open set that is smaller than U, but still covers $[a,b]$. Informally, every open superset is shrinkable: it is impossible to uniquely specify one to call "the smallest". Note also that

$$\bigcap_{n\in\mathbb{N}} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = [a,b]\,.$$

This shows that an infinite intersection of open sets may fail to be open.

C. Closed Sets and Closure

Definition. In a HTS $(X,\mathcal{T})$, a set $A \subseteq X$ is **closed** iff A^c is open.

Taking the set-complement in the definition above is different from taking the logical negation. A set can be "not open" at the same time that its complement is "not open", and in this case it is *neither open nor closed*. Even on the real line, all four possibilities below arise:

(1) the interval $(0,1)$ is open,

(2) the interval $[0,1]$ is closed,

(3) the interval $[0,1)$ is not open, and simultaneously not closed,

(4) the interval $(-\infty,+\infty)$ is open (obviously), and simultaneously closed (because its complement is $\emptyset$, which is open).

The simple relationship between open sets and closed sets opens the possibility that the whole axiomatic setup for Hausdorff Topological Spaces could have been expressed in terms of closed sets instead of open ones. Here are the details.

Proposition. *Given a nonempty set X, let $\mathcal{S}$ and $\mathcal{T}$ be subsets of $\mathcal{P}(X)$, related as follows:*

$$G \in \mathcal{T} \iff G^c \in \mathcal{S}, \qquad \text{i.e.,} \qquad F \in \mathcal{S} \iff F^c \in \mathcal{T}.$$

The following are equivalent:

(a) *$(X,\mathcal{T})$ is a HTS. (Every set in $\mathcal{T}$ is called "open".)*

(b) *The collection $\mathcal{S}$ has the four properties below; every set in $\mathcal{S}$ is called "closed":*

(i) *Both $\emptyset \in \mathcal{S}$ and $X \in \mathcal{S}$.*

(ii) *Any intersection of closed sets is a closed set.*

(iii) *Any union of finitely many closed sets is a closed set.*

(iv) *Whenever x_1 and x_2 are distinct points in X, there exist closed sets F_1, F_2 such that*

$$x_1 \in F_1 \setminus F_2, \quad x_2 \in F_2 \setminus F_1, \quad \text{and} \quad F_1 \cup F_2 = X.$$

Proof. The correspondence between properties (HTS1)–(HTS3) for $\mathcal{T}$ and properties (i)–(iii) for $\mathcal{S}$ follows directly from de Morgan's Laws. It remains to discuss how (HTS4) for $\mathcal{T}$ relates to property (iv) for $\mathcal{S}$.

In the following chain of statements involving general points $x_1, x_2 \in X$ and sets $G_1, G_2 \subseteq X$, each line is logically equivalent to the next one, subject only to the definitions $F_1 = G_2^c$ and $F_2 = G_1^c$:

$$\begin{aligned}
& x_1 \in G_1, && x_2 \in G_2, && \text{and} \quad G_1 \cap G_2 = \emptyset \\
\iff\ & x_1 \in G_1 \cap G_2^c, && x_2 \in G_2 \cap G_1^c, && \text{and} \quad (G_1^c \cup G_2^c)^c = \emptyset \\
\iff\ & x_1 \in G_2^c \cap (G_1^c)^c, && x_2 \in G_1^c \cap (G_2^c)^c, && \text{and} \quad G_2^c \cup G_1^c = X \\
\iff\ & x_1 \in F_1 \setminus F_2, && x_2 \in F_2 \setminus F_1, && \text{and} \quad F_1 \cup F_2 = X
\end{aligned}$$

(a)⇒(b): Assume that $(X, \mathcal{T})$ is a HTS. Given any distinct points $x_1, x_2 \in X$, there exist open sets G_1 and G_2 for which the first line above is true. The definitions of $F_1 = G_2^c$ and $F_2 = G_1^c$ then provide closed sets for which property (b)(iv) holds.

(b)⇒(a): Suppose (b) holds. Given any distinct points $x_1, x_2 \in X$, there exist closed sets F_1 and F_2 satisfying property (b)(iv), which is the bottom line in the chain of equivalences above. If we define $G_1 = F_2^c$ and $G_2 = F_1^c$, the top line of the chain confirms the conclusion of (HTS4) for these two points. ////

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Lemma. *In a HTS $(X, \mathcal{T})$, with $A \subseteq X$, TFAE:*

(i) *A is closed.*

(ii) *For every $x \notin A$, some neighbourhood $U \in \mathcal{N}(x)$ obeys $U \subseteq A^c$.*

Proof. Immediate from our similar lemma on open sets. ////

Lemma. *In a HTS $(X, \mathcal{T})$, every single-element set is closed.*

Proof. Let $K = \{x\}$. To show K^c is open, pick any $y \in K^c$. Then $y \neq x$, so there exist $U \in \mathbb{N}(x)$ and $V \in \mathbb{N}(y)$ s.t. $U \cap V = \emptyset$. In particular, $V \subseteq K^c$. So $y \in (K^c)^\circ$. This works for every $y \in K^c$, so every point of K^c is an interior point. This makes K^c an open set, as required. ////

Definition. Let A be a set in the HTS $(X, \mathcal{T})$. The **closure of** A is

$$\overline{A} = \big((A^c)^\circ \big)^c .$$

(Alternative notation: $\overline{A} = \operatorname{cl} A$.)

The definition implies monotonicity for this operation:

$$A \subseteq B \implies \overline{A} \subseteq \overline{B}.$$

Intuitively, $\overline{A}$ is the smallest closed superset of A, as expressed in parts (a)–(b) of the next result.

Prop. (a) *$\overline{A}$ is closed, and $\overline{A} \supseteq A$.*

(b) *If F is closed and $F \supseteq A$, then $F \supseteq \overline{A}$.*

(c) *A is closed if and only if $\overline{A} = A$.*

Proof. (a) As the complement of the open set $(A^c)^\circ$, the set $\overline{A}$ is closed. Also,

$$(A^c)^\circ \subseteq A^c \implies \left((A^c)^\circ\right)^c \supseteq (A^c)^c, \text{ i.e., } \overline{A} \supseteq A.$$

(b) Suppose F closed and $F \supseteq A$. Then F^c is open, so $F^c = (F^c)^\circ$, and

$$\begin{aligned} F \supseteq A &\implies F^c \subseteq A^c \\ &\implies F^c = (F^c)^\circ \subseteq (A^c)^\circ \\ &\implies F = (F^c)^c \supseteq \left((A^c)^\circ\right)^c = \overline{A}. \end{aligned}$$

(c) ($\Leftarrow$) Obvious, from (a). ($\Rightarrow$) If A is closed, then choosing $F = A$ in (b) gives $A \supseteq \overline{A}$. Hence, by (a), $A = \overline{A}$. ////

Example. If $a < b$ in $\mathbb{R}$, then $\overline{(a,b)} = \overline{[a,b)} = \overline{(a,b]} = \overline{[a,b]} = [a,b]$; $\overline{\mathbb{Q}} = \mathbb{R}$.

Practice. Earlier we saw that it is impossible to make sense of the phrase "smallest open superset of A". For practice, use the open interval $A = (0,1)$ to explain why the idea of a "largest closed subset of A" cannot be given a meaningful definition.

D. Boundary Points

Definition. Given a HTS $(X, \mathcal{T})$, let $A \subseteq X$. A point z in X is a **boundary point of** A iff

$$\forall G \in \mathcal{N}(z), \qquad \text{both } A \cap G \neq \emptyset \quad \text{and} \quad A^c \cap G \neq \emptyset.$$

The set of boundary points of A is denoted ∂A.

Remark. Boundary points may lie in either A or A^c. In fact, interchanging A and A^c in the definition above makes no logical difference, so we have

$$\partial A = \partial(A^c).$$

Example. In $\mathbb{R}$, $\partial(a,b) = \partial(a,b] = \partial[a,b) = \partial[a,b] = \{a,b\}$; $\partial\mathbb{Q} = \mathbb{R}$; $\partial\mathbb{Z} = \mathbb{Z}$; $\partial\mathbb{R} = \emptyset$. Notice that $\mathbb{Q} \subseteq \mathbb{R}$ but $\partial\mathbb{Q}$ is a strict superset of $\partial\mathbb{R}$. There is no "monotonicity" relation for the boundary operation.

For a set A in a HTS $(X, \mathcal{T})$,

(a) **$\partial A = \overline{A} \cap \overline{A^c}$.**

(b) *A is closed if and only if $\partial A \subseteq A$; indeed, $\overline{A} = A \cup \partial A$.*

(c) *A is open if and only if $\partial A \subseteq A^c$; indeed, $A^\circ = A \setminus \partial A$.*

Proof. Home Practice. ////

E. Limit Points and Isolated Points

Definition. Let A be a set in a HTS $(X, \mathcal{T})$. A point z in X is a **limit point of** A exactly when

$$\forall U \in \mathcal{N}(z), \quad (U \setminus \{z\}) \cap A \neq \emptyset.$$

The set of limit points for A is denoted A'. Other authors call these *cluster points* or *accumulation points* of A, and call A' the *derived set for A.*

Notes. (i) If $A \subseteq B$, then $A' \subseteq B'$.

(ii) $z \notin A' \iff \exists U \in \mathcal{N}(z) : (U \setminus \{z\}) \cap A = \emptyset.$

The terminology suggests a connection to limits. The next result shows that this is robust in the metric case, and at least partialy extendable to the general HTS context.

Proposition. *Let $(X, \mathcal{T})$ be a HTS, with $A \subseteq X$ and $x \in X$.*

(i) *The sequential property below guarantees that $x \in A'$:*

$(*)$ $\quad x = \lim_{n\to\infty} x_n$ *for some sequence (x_n) of distinct points, each in A.*

(ii) *If, in addition, the topology $\mathcal{T}$ comes from a metric, then every x in A' has property $(*)$.*

Proof. (i) Suppose $(*)$ holds. To show $x \in A'$, we check the definition. So let an arbitrary $U \in \mathcal{N}(x)$ be given. Then the definition of "$x_n \to x$" gives some $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. This gives an infinite number of distinct points in the set $U \cap A$. Passing to the subset $(U \setminus \{x\}) \cap A$ excludes at most one of these points (namely, x), so certainly $(U \setminus \{x\}) \cap A \neq \emptyset$.

(ii) Given any $x \in A'$, construct a suitable sequence as follows. Let $r_1 = 1$ and use the definition of $x \in A'$ to select some $x_1 \in \mathbb{B}(x; r_1) \cap A$. Note that $x_1 \neq x$, so we can define $r_2 = d(x_1, x)/2$ and be sure $r_2 > 0$. Since $x \in A'$, the set $\mathbb{B}(x; r_2) \cap A$ is nonempty, so we can select some x_2 in there; clearly $x_2 \neq x_1$. Let $r_3 = d(x_2, x)/2$. Choose $x_3 \in \mathbb{B}(x; r_3) \cap A$, and note that $x_3 \notin \{x_1, x_2\}$. Define $r_4 = d(x_3, x)/2$, and continue. This produces a sequence of distinct elements of A for which $d(x_n, x) < r_n \leq 2^{-n}$, so indeed $x_n \to x$, as required. ////

Remark. The careful formulation of the Proposition above opens the intriguing possibility that for some sets in some HTS situations, there are limit points that are not describable using sequences. This is correct, but subtle: the distinction will only be detectable in a topology that is fundamentally incompatible with a metric-based description.

Example. In $\mathbb{R}$, $a < b$ implies $(a, b)' = [a, b]' = [a, b]$; $\mathbb{Q}' = \mathbb{R}$ and $\mathbb{Z}' = \emptyset$.

Prop. *For any set G in a HTS $(X, \mathcal{T})$,*

$$G \text{ is open } \iff G \cap (G^c)' = \emptyset.$$

Proof. ($\Rightarrow$) Pick an arbitrary $x \in G$. Get some $U \in \mathcal{N}(x)$ such that $U \subseteq G$. Then $U \cap G^c = \emptyset$, so

$$\emptyset = (U \setminus \{x\}) \cap G^c.$$

That is, $x \notin (G^c)'$.

($\Leftarrow$) Pick any $x \in G$. By hypothesis, $x \notin (G^c)'$, so $\exists U \in \mathcal{N}(x)$ satisfying

$$\emptyset = (U \setminus \{x\}) \cap G^c = U \cap G^c. \qquad (\text{Recall } x \in G.)$$

That is, $U \subseteq (G^c)^c = G$. Since $x \in G$ is arbitrary, this shows G is open. ////

Cor. *For any set F in a HTS $(X, \mathcal{T})$,*

$$F \text{ is closed } \iff F \supseteq F'.$$

Proof. Apply the previous result to F^c:

$$F \text{ is closed } \iff F^c \text{ is open } \iff F^c \cap F' = \emptyset \iff F' \subseteq F. \qquad ////$$

Prop. *For any set A in a HTS $(X, \mathcal{T})$, the set A' is closed.*

Proof. Home Practice. ////

Theorem. *For any set A, one has $\overline{A} = A \cup A'$.*

Proof. ($\supseteq$) Let $F = \overline{A}$. Then F is closed, so $F \supseteq F'$. Also, $F \supseteq A$, so $F' \supseteq A'$. Hence $F \supseteq F' \supseteq A'$, and $F \supseteq A \cup A'$.

($\subseteq$) Let $C = A \cup A'$. An arbitrary $z \in C^c$ will obey both $z \notin A$ and $z \notin A'$, so $\exists U \in \mathcal{N}(z) \cap \mathcal{T}$ such that

$$\emptyset = (U \setminus \{z\}) \cap A = U \cap A, \quad \text{i.e.,} \quad U \subseteq A^c.$$

Hence $C^c \subseteq (A^c)^\circ$, giving $C \supseteq ((A^c)^\circ)^c = \overline{A}$. ////

Definition. For a set A in a HTS $(X, \mathcal{T})$, the collection of **isolated points** is $A \setminus A'$. I.e., x is an isolated point of A iff both

$$x \in A \quad \text{and} \quad \exists U \in \mathcal{N}(x) \,:\, A \cap U = \{x\}\,.$$

Example. In $(\mathbb{R}, \text{usual})$,

$\mathbb{Z}$ consists only of isolated points (i.e., $\mathbb{Z} = \mathbb{Z} \setminus \mathbb{Z}'$, because $\mathbb{Z}' = \emptyset$);

$\mathbb{Q}$ has no isolated points (i.e., $\mathbb{Q} \setminus \mathbb{Q}' = \mathbb{Q} \setminus \mathbb{R} = \emptyset$);

$A = [\mathbb{Q} \cap (-\infty, 0)] \cup \mathbb{N}$ has $A' = (-\infty, 0]$, so the isolated points of A form the set $\mathbb{N}$.

Terminology Summary. Given a set A in a HTS $(X, \mathcal{T})$,

$$\begin{aligned} A^c &= X \setminus A, \text{ the complement of } A \\ A^\circ &= \text{the interior of } A, \\ \overline{A} &= \text{the closure of } A, \\ A' &= \text{the limit points of } A, \\ \partial A &= \text{the boundary of } A, \\ A \setminus A' &= \text{the isolated points of } A. \end{aligned}$$

E.g., in $\mathbb{R}^k$, for the set $A = \mathbb{B}[0;1) \cup \{2\mathbf{e}_1\} = \left\{x \in \mathbb{R}^k \,:\, |x| < 1\right\} \cup \{(2, 0, \ldots, 0)\}$,

$$\begin{aligned} A^c &= \left\{x \in \mathbb{R}^k \,:\, |x| \geq 1\right\} \cap \left\{x \in \mathbb{R}^k \,:\, x \neq 2\mathbf{e}_1\right\}, \\ A^\circ &= \mathbb{B}[0;1), \\ \overline{A} &= \left\{x \in \mathbb{R}^k \,:\, |x| \leq 1\right\} \cup \{2\mathbf{e}_1\}, \\ A' &= \left\{x \in \mathbb{R}^k \,:\, |x| \leq 1\right\}, \\ \partial A &= \left\{x \in \mathbb{R}^k \,:\, |x| = 1\right\} \cup \{2\mathbf{e}_1\}, \\ A \setminus A' &= \{2\mathbf{e}_1\} = \{(2, 0, \ldots, 0)\}. \end{aligned}$$

F. Sequential Characterizations

In any HTS where the topology comes from a metric, every single one of the concepts and symbols introduced above can be described in terms of sequences. Earlier sections have presented sequential characterizations of open sets and limit points in metric spaces. Students are urged to formulate sequential descriptions also for closed sets, boundary points, and isolated points.

G. The sequence space ℓ^2

Recall the set ℓ^2 from HW07 Q2. A "point" x in ℓ^2 is a square-summable real sequence:

$$x = (x_1, x_2, \ldots),$$

and we have shown that the definitions

$$\langle x\,,\, y\rangle = \sum_{k=1}^{\infty} x_k y_k, \qquad \|x\| = \sqrt{\langle x\,,\, x\rangle}$$

make ℓ^2 similar in some ways to the finite-dimensional Euclidean spaces $\mathbb{R}^d$ $(d \in \mathbb{N})$. The similarity is strong enough to justify calling points in ℓ^2 "vectors" instead of "sequences". Taking $X = \ell^2$ is an excellent testing ground for our HTS concepts.

First, $d(x, y) = \|y - x\|$ defines a metric on $X = \ell^2$, so we have a metric space that's truly different from the familiar finite-dimensional Euclidean ones.

Second, ℓ^2 has another natural topology that is *Hausdorff but not metric.* That's the *weak topology*, defined next.

For any nonzero vector v in ℓ^2, the *hyperplane with normal* v is the linear subspace

$$\left\{y \in \ell^2 \,:\, \langle v\,,\, y\rangle = 0\right\}.$$

(Some would think of the subspace $\mathbb{R}v$ as a line in ℓ^2, and this hyperplane as its orthogonal complement, writing $(\mathbb{R}v)^\perp$.) Given any $x \in \ell^2$, the shifted set

$$x + \left\{y \in \ell^2 \,:\, \langle v\,,\, y\rangle = 0\right\} = \{x + y \,:\, \langle v\,,\, y\rangle = 0\} = \{w \,:\, \langle v\,,\, w - x\rangle = 0\}$$

is the *hyperplane through* x *with normal* v. Fatten this up to define a *slab centred on* x *with normal* v:

$$\Omega(x; v) = \{w \,:\, |\langle v\,,\, w - x\rangle\,| < 1\}.$$

Imagine moving away from the centre point x in the normal direction v, by choosing $w = x + tv$ for some parameter $t > 0$. You leave the slab at the instant when

$$1 = |\langle v\,,\, w - x\rangle\,| = |\langle v\,,\, tv\rangle\,| = \|tv\|\,\|v\| \qquad \Longleftrightarrow \qquad t = \frac{1}{\|v\|^2},$$

so the transition point obeys $\|w - x\| = \frac{1}{\|v\|}$. Thus the thickness of the slab is $2/\|v\|$: stretching the normal vector squeezes the slab.

We overload notation for slabs and say for any $x \in \ell^2$ and $V \subseteq \ell^2$,

$$\Omega(x; V) = \bigcap_{v \in V} \Omega(x; v) = \bigcap \{\Omega(x; v) \,:\, v \in V\}.$$

Here comes the weak topology: Collect into a family $\mathcal{T}$ all the subsets $G \subseteq \ell^2$ for which every point $x \in G$ has the property that $\Omega(x; V) \subseteq G$ for some *finite* set V. (Note that the finite set V typically depends on x; changing x might require a new V.)

Lemma. *For every $z \in \ell^2$ and every finite set $V \subseteq \ell^2$, one has $\Omega(z; V) \in \mathcal{T}$.*

Proof. Given a finite set V and point z, let $G = \Omega(z; V)$. Pick any $x \in G$. Then we know $|\langle x - z\,,\, v\rangle\,| < 1$ for each $v \in V$. Since V is a finite set, the following definition makes sense:

$$r = \max_{v \in V} |\langle x - z\,,\, v\rangle\,| < 1.$$

We must show that G contains a generalized slab centred at x. Let $\alpha = 1/(1 - r) > 0$ (noting $r + (1/\alpha) = 1$) and define $V_x = \{\alpha v \,:\, v \in V\}$. The vectors in V_x are just stretched versions of the vectors in V, so in particular V_x is a finite set. Let's show that $\Omega(x; V_x) \subseteq \Omega(z; V)$. To prove this, pick any $y \in \Omega(x; V_x)$ and test the inequalities required for $y \in \Omega(z; V)$. This combines the add-subtract trick with the multiply-divide trick:

$$\forall v \in V, \qquad |\langle y - z\,,\, v\rangle\,| \le \frac{1}{\alpha}|\langle y - x\,,\, \alpha v\rangle\,| + |\langle x - z\,,\, v\rangle\,| < \frac{1}{\alpha} + r = 1,$$

as required. ////

H. Subspaces; Relative Topologies; Bases

Definition. Let $(X, \mathcal{T})$ be a HTS. For any subset $Y \subseteq X$, the collection

$$\mathcal{T}_Y = \{G \cap Y \,:\, G \in \mathcal{T}\}\,.$$

makes $(Y, \mathcal{T}_Y)$ into a HTS all by itself. (Practice: Check axioms (HTS1)–(HTS4).) It's typical to call $(Y, \mathcal{T}_Y)$ a **(topological) subspace of** $(X, \mathcal{T})$**.** (There is no simple relationship with the concept of "subspace" in linear algebra.)

In the context above, it's hard to completely forget about X, so we must be especially careful with the words we use. For example, we will call a set $A \in \mathcal{T}_Y$ **open relative to** Y. To see why this is important, let X be the real line with the usual topology for $\mathcal{T}$, and consider the subset $Y = [-1, 1]$. The set $A = (0, 1]$ is open relative to Y, because $A = (0, 2) \cap Y$ belongs to $\mathcal{T}_Y$ ... but clearly A is not open in the original sense ("relative to X").

If the subset $Y \subseteq X$ happens to be open to start with, then $\mathcal{T}_Y \subseteq \mathcal{T}$. In this case a subset of Y is open relative to $\mathcal{T}_Y$ iff it is open in the original sense from X. Thus the terms "open relative to Y" and simply "open" mean the same thing. But, as indicated above, if we start with a Y that is not open in X, extra care is needed.

Example. Let $X = \mathbb{R}$. If $Y = \mathbb{Q}$, the set $W = (0, 1) \cap \mathbb{Q}$ is open relative to Y, but it is not open in X.
Or, let $X = \mathbb{R}^2$ and let $Y = \mathbb{R} \times \{0\} = \{(x, 0) : x \in \mathbb{R}\}$. The segment $I = \{(x, 0) : 0 < x < 1\}$ is open relative to Y, but not open in X. ////

A topological base. Given any nonempty set X and a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$, try using sets in $\mathcal{B}$ to define neighbourhoods and reverse-engineer a topology. That is, for each x in X, make a speculative definition that looks very familiar:

$$\mathcal{N}(x) = \{S \subseteq X \,:\, \exists B \in \mathcal{B} \text{ with } x \in B,\ B \subseteq S\}\,.$$

Then declare a set $G \subseteq X$ to be "open" if and only if either $G = \emptyset$ or $G \in \mathcal{N}(x)$ holds for each $x \in G$. We know and trust this linkage between open sets and neighbourhoods in any HTS. What properties do we need from the set $\mathcal{B}$ to make this approach succeed? The answer is embedded in the next result.

Proposition. *Given a nonempty set X, suppose $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfies conditions* (a)–(c) *below. Then the construction above defines a Hausdorff topology $\mathcal{T}$ on X. In this case, the set $\mathcal{B}$ is called a* **base** *for $\mathcal{T}$.*

(a) $\bigcup \mathcal{B} = X$ *[every point of X belongs to at least one set B in $\mathcal{B}$];*

(b) whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists some $B \in \mathcal{B}$ satisfying $x \in B \subseteq B_1 \cap B_2$; and

(c) whenever $x_1, x_2 \in X$ obey $x_1 \neq x_2$, there exist $B_1, B_2 \in \mathcal{B}$ satisfying $x_1 \in B_1$, $x_2 \in B_2$, and $B_1 \cap B_2 = \emptyset$.

Examples. In $\mathbb{R}^k$, let $\mathcal{B}$ denote the set of balls $\mathbb{B}[p;r)$ for which the centre point p lies in $\mathbb{Q}^k$ and the radius $r > 0$ is rational. This is a base for the usual topology on $\mathbb{R}^k$. It is interesting because $\mathcal{B}$ is a *countable* collection of sets. A topological space for which a countable base exists is called "second-countable": see "Second-countable space" on Wikipedia if you would like to know more.