

VI. Sequences

UBC M320 Lecture Notes by Philip D. Loewen

A. Background

reading

Please see Canvas for a short writeup justifying the Cauchy-Schwarz inequality and the triangle inequality in $\mathbb{R}^d$.

B. Sequences and Convergence

2026-10-02

The next result is easy, but sometimes useful.

Lemma. For x and L in $\mathbb{R}^d$, TFAE:

- (a) $x = L$.
- (b) $\forall \varepsilon > 0, |x - L| < \varepsilon$.

Proof. (a $\Rightarrow$ b) Trivial.

(b $\Rightarrow$ a) Use contraposition. Given x and L ,

$$\neg(\text{a}) \implies x \neq L \implies \varepsilon = |x - L| > 0 \implies \exists \varepsilon > 0 \text{ s.t. } |x - L| \geq \varepsilon \implies \neg(\text{b}).$$

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Definition. (Sequences) Let X be a set. A **sequence in** X is a function $x: \mathbb{N} \rightarrow X$. We often write x_n for the value $x(n)$, and denote the sequence itself by $(x_n)_{n=1}^\infty$. [Notation resembles “a vector with $\aleph_0$ components”.] The **range** of the sequence $(x_n)_{n=1}^\infty$ is an (unordered) subset of X , namely, $\{x_n : n \in \mathbb{N}\}$.

Remark. The real sequence $x_n = (-1)^n \forall n$ defines infinitely many *symbols* $x_1, x_2, x_3, \dots$, but its range is just the two-element set $\{x_n : n \in \mathbb{N}\} = \{-1, 1\}$.

Definition. (Limits) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^d$. To say that **the sequence** (x_n) **converges** means that there exists $L \in \mathbb{R}^d$ such that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, |x_n - L| < \varepsilon. \quad (*)$$

In this case we say “ (x_n) converges to L ” and write

$$x_n \rightarrow L \text{ as } n \rightarrow \infty, \quad \text{or} \quad x_n \xrightarrow{n \rightarrow \infty} L.$$

Intuition/Terminology. In $(*)$, the number $\varepsilon > 0$ is an “error tolerance”. The inequality $|x_n - L| < \varepsilon$ holds exactly when x_n lies in an open ball of radius ε centred at L . In the scalar case ($d = 1$) it is equivalent to

$$-\varepsilon < x_n - L < \varepsilon, \quad \text{i.e.,} \quad x_n \in (L - \varepsilon, L + \varepsilon).$$

The quantifiers on N and n say that this inclusion holds *for all n sufficiently large*, or *for all but finitely many n* . The quantifier on ε requires the statement above to be valid for every $\varepsilon > 0$.

Remark. The order of the quantifiers (“ $\forall, \exists$ ”) in the definition is critical: for every $\varepsilon > 0$ something must happen, but *how* it happens may vary from one ε -value to another. In particular, the choice of N almost always *depends on* ε . Consider reading the symbol $\forall$ as “for each”.

Proposition. *Convergent sequences have unique limits. That is, if a sequence $(x_n)_n$ in $\mathbb{R}^d$ satisfies both $x_n \rightarrow L_1$ and $x_n \rightarrow L_2$, then $L_1 = L_2$.*

Proof. In the situation described above, let any $\varepsilon > 0$ be given. Define $\varepsilon' = \varepsilon/2$ and use the definition of convergence twice, with tolerance ε' both times. Knowing $x_n \rightarrow L_1$, there must be some $N_1 \in \mathbb{N}$ such that every $n > N_1$ makes $|x_n - L_1| < \varepsilon'$. Similarly, knowing $x_n \rightarrow L_2$ implies that there exists $N_2 \in \mathbb{N}$ such that every $n > N_2$ makes $|x_n - L_2| < \varepsilon'$. Now choose any $n > \max\{N_1, N_2\}$. This activates both inequalities above, leading to

$$|L_1 - L_2| = |(L_1 - x_n) + (x_n - L_2)| \leq |x_n - L_1| + |x_n - L_2| < \varepsilon' + \varepsilon' = \varepsilon.$$

In short, for each $\varepsilon > 0$ we have $|L_1 - L_2| < \varepsilon$. According to the easy Lemma above, this forces $L_1 = L_2$. ////

Knowing that limits are unique justifies introducing the notation

$$\lim_{n \rightarrow \infty} x_n.$$

This compound symbol stands for L whenever $x_n \rightarrow L$. Later we will allow the symbolic values $+\infty$ and $-\infty$ in place of L . We also tolerate the string “DNE” as a descriptive outcome, when we need a symbolic shorthand to say the sequence (x_n) diverges but neither $-\infty$ nor $+\infty$ is appropriate. Unlike those extended real numbers, “DNE” cannot participate in any relation involving the symbol “=”.

Definition. (Divergence) “Diverges” is the opposite of “converges.” That is,

$$\begin{aligned} (x_n) \text{ diverges} &\Leftrightarrow \neg [(x_n) \text{ converges}] \\ &\Leftrightarrow \neg [\exists L \in \mathbb{R}^d : \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, |x_n - L| < \varepsilon] \quad (\text{DNE}) \\ &\Leftrightarrow \forall L \in \mathbb{R}^d, \exists \varepsilon > 0 : \forall N \in \mathbb{N}, \exists n > N : |x_n - L| \geq \varepsilon. \end{aligned}$$

(Think about how to express this symbolic assertion in simple words.)

Home Practice. Show that the order of the quantifiers in (*) really matters by proving that The Following Are Equivalent (TFAE):

- (i) $\exists N \in \mathbb{N} : \forall \varepsilon > 0, \forall n > N, |x_n - L| < \varepsilon$. [NOT our definition of a limit.]
- (ii) $x_n = L$ for all but at most finitely many $n \in \mathbb{N}$. [NOT our intuitive concept of a limit.]

Example. Simple examples in $(\mathbb{R}, |\cdot|)$:

(a) $x_n = 1/n$ converges to $L = 0$.

Pf: Given $\varepsilon > 0$, choose any integer $N \geq 1/\varepsilon$: then for all $n > N \geq 1/\varepsilon$, one has $|x_n - 0| = 1/n < \varepsilon$.

(b) For any real $M \neq 0$ and $p > 0$, $x_n = M/n^p$ converges to $L = 0$.

Pf: Given $\varepsilon > 0$, choose any integer $N \geq \left(\frac{|M|}{\varepsilon}\right)^{1/p}$: then for all $n > N$, one has

$$n^p > \frac{|M|}{\varepsilon} \implies \frac{1}{n^p} < \frac{\varepsilon}{|M|} \implies |x_n - 0| = \frac{|M|}{n^p} < \varepsilon.$$

(c) $x_n = 1$ converges to $L = 1$.

Pf: Given $\varepsilon > 0$, choose $N = 320$. Then $n > N \Rightarrow |x_n - 1| = 0 < \varepsilon$.

Note: Any $N \in \mathbb{N}$ (not just 320) will work. The choice $N = 1$ would be most “efficient”, but the definition contains no criterion for efficiency.

Remarks. 1. Convergence statement (*) is self-contained: variable quantities ε and N are internal, not visible from outside. It does not make sense to ask, “What is the value of ε in line (*)?” [Analogy: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called *increasing* when “ $a < b \Rightarrow f(a) < f(b)$ ”. Given an increasing function, it is meaningless to ask, “Find a to six sig figs.”]

2. To confirm (*), imagine constructing a **function** $N = N(\varepsilon)$ that is

(i) Defined for all $\varepsilon > 0$ [both very small and very large],

(ii) Consistent with $\forall n > N(\varepsilon), |x_n - L| < \varepsilon$.

In most interesting cases, the number $N(\varepsilon)$ emerges from a process of logical argumentation rather than simple algebra. It’s overly simplistic to expect $N(\varepsilon)$ to be captured by a simple formula; the point here is the intuitive idea of what we are trying to achieve.

3. The only criterion for finding a suitable $N(\varepsilon)$ is *existence*, not efficiency. A typical strategy is to invent a simple overestimate of $|x_n - L|$ and choose N to make that small. (“To guarantee that $|x_n - L|$ is small, it would be *more than enough* to make this simpler thing small.”)

4. Shrinking the value of $\varepsilon > 0$ makes the inequality $|x_n - L| < \varepsilon$ a more demanding restriction on the elements x_n . So it’s typical for $N(\varepsilon)$ to grow as ε shrinks.

5. Insisting that $N \in \mathbb{N}$ in the definition is not a serious limitation: allowing $N \in \mathbb{R}$ gives a concept with exactly the same meaning. [Exercise: Writing the definition to require N to be a prime number instead of a generic positive integer also gives exactly the same concept.] So we’re not going to enforce the distinction on homework and tests.

Example. (More Serious Examples)

(a) $x_n = \sin(n)/(n^5 + n^4 + n^3 + n^2 + n + 1)$ converges to $L = 0$.

Pf: For any $n \in \mathbb{N}$,

$$|x_n - L| = \left| \frac{\sin(n)}{n^5 + n^4 + n^3 + n^2 + n + 1} \right| \leq \left| \frac{\sin(n)}{0 + 0 + 0 + 0 + n + 0} \right| \leq \frac{1}{n},$$

so any integer $N \geq 1/\varepsilon$ will satisfy definition (*). Alternatively, each $n \in \mathbb{N}$ obeys

$$|x_n - L| = \left| \frac{\sin(n)}{n^5 + n^4 + n^3 + n^2 + n + 1} \right| \leq \left| \frac{\sin(n)}{n^5 + 0 + 0 + 0 + 0 + 0} \right| \leq \frac{1}{n^5},$$

so any integer $N \geq 1/\varepsilon^{1/5}$ will also satisfy definition (*).

Discussion: In terms of the intuitive discussion above, both $N_1(\varepsilon) = 1/\varepsilon$ and $N_2(\varepsilon) = 1/\varepsilon^{1/5}$ will verify definition (*). As ε shrinks toward 0, the values of $N_1(\varepsilon)$ grow much faster than the values of $N_2(\varepsilon)$:

$$\varepsilon = 10^{-5} \implies N_1(\varepsilon) = 100,000, \quad N_2(\varepsilon) = 10.$$

This provides an emphatic concrete illustration of the general principles above: existence is our only concern; efficiency is a problem for some other time.

(b) [Not shown in class.] $x_n = (-1)^n$ diverges (or “has no limit”).

Pf: We confirm (DNE) directly. Given any L in $\mathbb{R}$, consider $\varepsilon = 1$: fix any $N \in \mathbb{N}$. Then there exist an odd integer $n_o > N$ and an even integer $n_e > N$, for which the triangle inequality gives

$$2 = |x_{n_e} - x_{n_o}| = |(x_{n_e} - L) + (L - x_{n_o})| \leq |x_{n_e} - L| + |x_{n_o} - L|.$$

This implies that either $|x_{n_e} - L| \geq 1 = \varepsilon$ or else $|x_{n_o} - L| \geq 1 = \varepsilon$. /////

(c) The sequence $x_n = \frac{n^2 - 320n^{3/2}}{2n^2 - 801}$ converges to $L = \frac{1}{2}$.

Pf: Given an arbitrary $\varepsilon > 0$, choose the smallest integer $N \geq \max \left\{ 30, \left(\frac{750}{\varepsilon} \right)^2 \right\}$.

Then every $n > N$ satisfies $n^2 > 30^2 = 900$, so

$$(\dagger) \quad 2n^2 - 801 = n^2 + (n^2 - 801) > n^2 \quad \text{and} \quad (\ddagger) \quad \sqrt{n} > \frac{750}{\varepsilon},$$

and these inequalities (among others) imply

$$\begin{aligned}
 |x_n - L| &= \left| \frac{n^2 - 320n^{3/2}}{2n^2 - 801} - \frac{1}{2} \right| \\
 &= \frac{|(2n^2 - 640n^{3/2}) - (2n^2 - 801)|}{2|2n^2 - 801|} \\
 &\leq \frac{640n^{3/2} + 801}{2|2n^2 - 801|} && \text{(triangle inequality)} \\
 &< \frac{640n^{3/2} + 801n^{3/2}}{2n^2} && \text{(from } (\dagger), \text{ and } 1 < n^{3/2}) \\
 &< \frac{750}{\sqrt{n}} && (640 + 801 < 1500) \\
 &< \frac{750}{750/\varepsilon} && \text{(from } (\dagger)) \\
 &= \varepsilon,
 \end{aligned}$$

as required. /////

Remarks. Loose overestimates like “ $640 + 801 < 1500$ ” and “ $801 \leq 801n^{3/2}$ ” embedded above are fair play: they are admittedly sloppy, but not wrong. The point is simply to show **existence** of some $N = N(\varepsilon)$ that satisfies the definition, **not** to find the best one. The point at which an inequality becomes too sloppy is when it changes the dominant order of growth of the multinomial where it appears. Think about why the argument that succeeded with the sloppy inequality “ $801 \leq 801n^{3/2}$ ”, would break down utterly if we had opted for “ $801 \leq 801n^2$ ” instead.

2026-10-07

$$(*) : \quad x_n \rightarrow L \text{ (in } \mathbb{R}^d) \iff \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, |x_n - L| < \varepsilon.$$

Proposition. One has $n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$.

Lemma. For any real $x \geq 0$,

$$(1+x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2 \quad \forall n \in \mathbb{N}.$$

Proof. (Lemma) Recall the Binomial Theorem: for any $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for } 0 \leq k \leq n.$$

(Also $0! = 1$.) This implies that for each $z \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2}z^2 + \frac{n(n-1)(n-2)}{6}z^3 + \dots + nz^{n-1} + z^n.$$

Given a real $x \geq 0$, choosing $z = x$ gives the stated result. (Every term dropped from the right side is nonnegative.) ////

Proof. (Prop) To show $x_n = n^{1/n}$ converges to $L = 1$, it suffices to show that $y_n = n^{1/n} - 1$ converges to 0. Clearly $y_n \geq 0$, so the Lemma gives

$$n = (1 + y_n)^n \geq \frac{n(n-1)}{2} y_n^2.$$

When $n \geq 2$, we have $n-1 \geq n/2$, so the inequality above gives

$$y_n \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{n/2}} = \frac{2}{\sqrt{n}}.$$

Given any $\varepsilon > 0$, the inequality $y_n < \varepsilon$ can be guaranteed by insisting on both $n \geq 2$ and $2/\sqrt{n} < \varepsilon$, i.e., $n > 4/\varepsilon^2$. So we choose any integer

$$N \geq \max \left\{ 2, \frac{4}{\varepsilon^2} \right\}.$$

Every $n > N$ will give $|n^{1/n} - 1| < \varepsilon$, as required. ////

C. Monotone Sequences

Proposition. Any real sequence (x_n) with bounded range that is eventually non-decreasing, i.e.,

$$\exists N_0 \in \mathbb{N} : \forall n > N_0, x_n \leq x_{n+1},$$

must converge. The limit is $L = \sup \{x_n : n > N_0\}$.

Proof. Given $\varepsilon > 0$ and L as above, consider $\gamma = L - \varepsilon$: since $\gamma < L$, there must be some $N > N_0$ such that $x_N > \gamma$. This N works in definition (*), because every $n > N$ makes

$$L - \varepsilon < x_N \leq x_n \leq L < L + \varepsilon.$$

The first inequality comes from our choice of N ; the second holds because the sequence is nondecreasing for subscripts exceeding N_0 (which these subscripts do); the third holds because L is an upper bound for a set including the point of interest; and the fourth is obvious. /////

Practice. There is an analogous result for real sequences that are eventually nonincreasing. State and prove this result.

Extended Values. For an increasing sequence like $x_n = n$ that has no upper bound, it's tempting to write $x_n \rightarrow +\infty$ as $n \rightarrow \infty$. Let's write a definition to legitimize this.

Definition. Say that a real-valued sequence $(x_n)_n$ *diverges to* $+\infty$ exactly when

$$\forall R \in \mathbb{R}, \exists N \in \mathbb{N} : \forall n > N, x_n > R.$$

Express this with the notation

$$x_n \rightarrow \infty \text{ (as } n \rightarrow \infty) \quad \text{or} \quad \lim_{n \rightarrow \infty} x_n = +\infty.$$

Practice. Write an analogous definition for $x_n \rightarrow -\infty$ as $n \rightarrow \infty$.

Remarks. 1. We will strictly defend the verb “to converge”: this is used only in cases where the limit is a real number (or, more generally, a vector in $\mathbb{R}^d$).

2. Permitted extended-real values for the compound symbol “ $\lim_{n \rightarrow \infty} x_n$ ” are $-\infty$, $+\infty$, or a real number. When none of these values is appropriate, we say the limit *does not exist*, and use the abbreviation DNE. This is not even an extended number, so we will object to any assertion that something *equals* DNE. We just use it as a prose-like abbreviation, as shown below:

$$\begin{aligned} \lim_{n \rightarrow \infty} (n - n^2) &= -\infty, & \lim_{n \rightarrow \infty} \frac{cn + k}{n + 2} &= c, & \lim_{n \rightarrow \infty} \frac{2^n}{n} &= +\infty, \\ \lim_{n \rightarrow \infty} (-1)^n &\text{DNE.} \end{aligned}$$

3. We already allow extended real values $\pm\infty$ for the supremum and infimum, and these are fully compatible with the setup here. The next result is a versatile extension of the previous Proposition.

Proposition. (a) For any real sequence (x_n) that is eventually nondecreasing, i.e.,

$$\exists N_0 \in \mathbb{N} : \forall n > N_0, x_n \leq x_{n+1},$$

one has $x_n \rightarrow L$ as $n \rightarrow \infty$, where $L = \sup \{x_n : n > N_0\}$. (The sequence converges if $L \in \mathbb{R}$ and diverges if $L = +\infty$.)

(b) For any real sequence (x_n) that is eventually nonincreasing, i.e.,

$$\exists N_0 \in \mathbb{N} : \forall n > N_0, x_n \geq x_{n+1},$$

one has $x_n \rightarrow L$ as $n \rightarrow \infty$, where $L = \inf \{x_n : n > N_0\}$. (The sequence converges if $L \in \mathbb{R}$ and diverges if $L = -\infty$.)

Proof. Exercise. ////

D. Limit Laws

Theorem. If the $\mathbb{R}^d$ -valued sequences (x_n) , (y_n) converge, with $L = \lim_n x_n$ and $M = \lim_n y_n$, and if $c \in \mathbb{R}$, then

(a) $\lim_n (cx_n) = cL,$

(b) $\lim_n (x_n + y_n) = L + M.$

Proof. (a) For any $n \in \mathbb{N}$,

$$|cx_n - cL| = |c||x_n - L| \leq (1 + |c|)|x_n - L|.$$

So given any $\varepsilon > 0$, define $\varepsilon' = \varepsilon/(1 + |c|)$ and appeal to the definition of $x_n \rightarrow L$ using ε' as the tolerance: this yields some $N \in \mathbb{N}$ so large that the desired conclusion holds, i.e.,

$$n > N \implies |cx_n - cL| \leq (1 + |c|)|x_n - L| < (1 + |c|) \left(\frac{\varepsilon}{1 + |c|} \right) = \varepsilon.$$

(b) For any $n \in \mathbb{N}$, we have

$$|(x_n + y_n) - (L + M)| = |(x_n - L) + (y_n - M)| \leq |x_n - L| + |y_n - M|.$$

So given any $\varepsilon > 0$, define $\varepsilon' = \varepsilon/2$ and appeal to the definitions of $x_n \rightarrow L$ and $y_n \rightarrow M$ to get some $N_1, N_2 \in \mathbb{N}$ such that $n > N_1$ guarantees $|x_n - L| < \varepsilon'$ and $n > N_2$ guarantees $|y_n - M| < \varepsilon'$. Construct $N = \max \{N_1, N_2\}$: then every $n > N$ obeys

$$|(x_n + y_n) - (L + M)| \leq |x_n - L| + |y_n - M| < \varepsilon' + \varepsilon' = \varepsilon,$$

as required. ////

Observations. 1. For a sequence (x_n) in $\mathbb{R}^d$, one has

$$x_n \rightarrow L \iff (x_n - L) \rightarrow 0 \iff |x_n - L| \rightarrow 0.$$

(Compare the definitions.)

2. The letters “ n ”, “ ε ”, “ N ” in the definition are strictly internal (think of a stand-alone procedure in computer programming): that logical T/F statement is equivalent to

$$(*_R) : \quad \forall \rho > 0, \exists R \in \mathbb{N} : \forall r > R, |x_r - L| < \rho.$$

In particular, the N -vs- ε relationship for one convergence statement is typically completely unrelated to the R -vs- ρ relationship for another.

Lemma. Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}^d$; denote $L = \lim_{n \rightarrow \infty} x_n$. Then

- (a) (x_n) is bounded, i.e.,

$$\exists R > 0 : \forall n \in \mathbb{N}, |x_n| \leq R.$$

- (b) If $L \neq 0$, then (x_n) is “eventually bounded away from zero”, i.e.,

$$\exists r > 0, N \in \mathbb{N} : \forall n > N, |x_n| > r.$$

Remark. The converse of (a) is false: some bounded sequences, like $(-1)^n$ in $\mathbb{R}$, are not convergent. The contrapositive of (a) says “not bounded implies not convergent”.

Proof. (a) Take $\varepsilon = 1$ in the definition of $x_n \rightarrow L$: it guarantees that $\exists N$ such that

$$|x_n - L| < 1 \quad \forall n > N.$$

Hence for all $n \in \mathbb{N}$,

$$|x_n - L| \leq \max \{|x_1 - L|, |x_2 - L|, \dots, |x_N - L|, 1\} \stackrel{\text{def}}{=} R_0.$$

Since $|x_n| = |L + (x_n - L)| \leq |L| + |x_n - L|$, this gives

$$|x_n| \leq |L| + R_0 \stackrel{\text{def}}{=} R,$$

as required.

- (b) The statement is true for any $r > 0$ such that $r < |L|$. Indeed, pick any such r . Apply definition of $x_n \rightarrow L$ to $\varepsilon = |L| - r > 0$. It gives some $N \in \mathbb{N}$ such that, for each $n > N$,

$$|L| - r = \varepsilon > |x_n - L| = |L - x_n| \geq |L| - |x_n|.$$

Thus we have (for each $n > N$) the inequality $-r > -|x_n|$, i.e., $|x_n| > r$. ////

$$(*) : \quad x_n \rightarrow L \text{ (in } \mathbb{R}^d) \iff \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, |x_n - L| < \varepsilon.$$

In proving a convergence statement like $(*)$, the main effort typically goes into finding reasons for which the discrepancy $|x_n - L|$ is small. Packaging those reasons into the form required by $(*)$ can often be entrusted to a result of the form shown next. Both of the following results can be referred to as versions of the Squeeze Theorem.

Proposition (Squeeze Theorem). *Suppose real-valued sequences (a_n) , (x_n) , and (b_n) are given, and $\exists N \in \mathbb{N} : \forall n > N, a_n \leq x_n \leq b_n$.*

- (i) *If both $a_n \rightarrow L$ and $b_n \rightarrow L$, with $L \in \mathbb{R}$, then $x_n \rightarrow L$.*
- (ii) *If $a_n \rightarrow +\infty$, then $x_n \rightarrow +\infty$.*
- (iii) *If $b_n \rightarrow -\infty$, then $x_n \rightarrow -\infty$.*

Proof. (i) Let $\varepsilon > 0$ be given. Use the known convergence facts for (a_n) and (b_n) to get some $N_a, N_b \in \mathbb{N}$ such that

$$\begin{aligned} \forall n > N_a, \quad L - \varepsilon < a_n < L + \varepsilon, \\ \forall n > N_b, \quad L - \varepsilon < b_n < L + \varepsilon. \end{aligned}$$

Then let $N_s \in \mathbb{N}$ guarantee activation of the sandwich condition:

$$\forall n > N_s, \quad a_n \leq x_n \leq b_n.$$

Let $N = \max\{N_a, N_b, N_s\}$. Chain together the properties above to see that every $n > N$ will give an x_n that meets the required tolerance:

$$L - \varepsilon < a_n \leq x_n \leq b_n < L + \varepsilon, \quad \text{so} \quad |x_n - L| < \varepsilon.$$

(ii)(iii) Exercise. (Outcomes (ii) and (iii) require only one auxiliary sequence. Imagine $b_n = x_n$ or $a_n = x_n$ to streamline the sandwich condition in these respective cases, but don't bother saying all this when applying the result.) /////

Corollary. *For $L \in \mathbb{R}^d$ and a sequence (x_n) , TFAE:*

- (a) $x_n \rightarrow L$ as $n \rightarrow \infty$,
- (b) *some real-valued sequence (r_n) for which $r_n \rightarrow 0$ as $n \rightarrow \infty$ eventually dominates $|x_n - L|$, i.e.,*

$$\exists N \in \mathbb{N} : \forall n > N, |x_n - L| \leq r_n.$$

Proof. (a $\Rightarrow$ b) Trivial: Define $r_n = |x_n - L|$.

(b $\Rightarrow$ a) By hypothesis, we have

$$0 \leq |x_n - L| \leq r_n, \quad \forall n > N.$$

So the variant of the Squeeze Theorem above gives $|x_n - L| \rightarrow 0$. This is equivalent to $x_n \rightarrow L$ in $\mathbb{R}^d$. /////

Theorem. If the $\mathbb{R}^d$ -valued sequences (x_n) , (y_n) converge, with $L = \lim_n x_n$ and $M = \lim_n y_n$, then

$$(a) \lim_n (x_n \bullet y_n) = L \bullet M,$$

$$(b) \text{ if } d = 1 \text{ and } M \neq 0, \text{ with all } y_n \neq 0, \text{ then } \lim_n \left(\frac{1}{y_n} \right) = \frac{1}{M}.$$

Proof. (Sketch) Combine the Squeeze Theorem and the rules for sums and scalar multiples, taking r and R from the previous Lemma.

The key inequality for (a), valid for all n sufficiently large, is

$$\begin{aligned} |x_n \bullet y_n - L \bullet M| &= |(x_n - L + L) \bullet y_n - L \bullet M| \\ &\leq |x_n - L| |y_n| + |L| |y_n - M| \\ &\leq R |x_n - L| + |L| |y_n - M|. \end{aligned}$$

As $n \rightarrow \infty$, the right side converges to 0.

For (b), we use (for all n sufficiently large)

$$\left| \frac{1}{y_n} - \frac{1}{M} \right| = \frac{|M - y_n|}{|M y_n|} \leq \frac{1}{r |M|} |y_n - M|.$$

Again, sending $n \rightarrow \infty$ drives the right side to 0, and the result follows. ////

Example. For each $r > 0$, $\lim_{n \rightarrow \infty} r^{1/n} = 1$.

Proof. Suppose $r \geq 1$. Then there exists some $N \in \mathbb{N}$ such that $r < N$. Whenever $n > N$, we have $1 \leq r < n$ so $1 \leq r^{1/n} < n^{1/n}$. Send $n \rightarrow \infty$ and apply the Squeeze Theorem with $a_n = 1$, $x_n = r^{1/n}$, and $b_n = n^{1/n}$.

Next, suppose $r \in (0, 1)$. Let $R = 1/r > 1$. Then $R^{1/n} \rightarrow 1$ by the previous case. Therefore we can define $y_n = R^{1/n}$ and apply part (b) above with $M = 1$: it follows that $1/y_n = 1/(R^{1/n}) = (1/R)^{1/n} = r^{1/n}$ converges to $1/M = 1$. ////

The Vector Case. Any sequence $(x_n)_n$ in $\mathbb{R}^d$ has d “component sequences” in $\mathbb{R}$. The first one is $(\mathbf{e}_1 \bullet x_n)_n$; the p -th one is $(\mathbf{e}_p \bullet x_n)_n$. E.g., for the sequence x in $\mathbb{R}^3$ defined by $x_n = (n^{-1} \sin n, \frac{n+1}{n}, n^{1/n})$, there are 3 component sequences:

$$\mathbf{e}_1 \bullet x_n = n^{-1} \sin n, \quad \mathbf{e}_2 \bullet x_n = \frac{n+1}{n}, \quad \mathbf{e}_3 \bullet x_n = n^{1/n}.$$

We have $x_n \rightarrow (0, 1, 1)$ by the following result.

Theorem. A sequence in $\mathbb{R}^d$ converges iff each of its d component sequences converges. In this case, componentwise evaluation of the limit produces the correct result.

Proof. Invent the notation $\underline{d} = \{1, 2, 3, \dots, d\}$. The result is an immediate consequence of the Squeeze Theorem. Here we sketch the reasons.

($\Rightarrow$) Let (x_n) be a convergent sequence in $\mathbb{R}^d$; define $L = \lim x_n$. For any fixed $p \in \underline{d}$, every n obeys

$$|(\mathbf{e}_p \bullet x_n) - (\mathbf{e}_p \bullet L)| = |\mathbf{e}_p \bullet (x_n - L)| \leq 1 \cdot |x_n - L|.$$

($\Leftarrow$) Suppose that for each $p \in \underline{d}$, the component sequence $(\mathbf{e}_p \bullet x_n)$ converges; define $L_p = \lim(\mathbf{e}_p \bullet x_n)$ and then form $L = (L_1, \dots, L_d) \in \mathbb{R}^d$. Resolve the vector difference $x_n - L$ into components, and use the triangle inequality to get

$$|x_n - L| = \left| \sum_{p=1}^d [\mathbf{e}_p \bullet (x_n - L)] \mathbf{e}_p \right| \leq \sum_{p=1}^d |\mathbf{e}_p \bullet (x_n - L)|.$$

The right side is the sum of d sequences, each of which converges to 0 by hypothesis. By the Sum Rule, the right side converges to 0. According to the Squeeze Theorem, the left side does too. ////

Three Viewpoints on Convergence. One can approach the definition of “ $x_n \rightarrow L$ ” given in line (*) above in three distinct ways:

1. As a prover. If your job is to justify the statement “ $x_n \rightarrow L$ ”, then you must find a convincing reason why, for each (arbitrary) $\varepsilon > 0$, the favourable outcome described in (*) actually occurs. The first line in such an argument is typically, “Let any $\varepsilon > 0$ be given.”
2. As a user. In a context where the statement “ $x_n \rightarrow L$ ” is *given*, you can use it to produce other ingredients for use in logical constructions. In particular, if (*) is known to be true, then you can exploit it by choosing any $\varepsilon > 0$ that will meet your needs, and feeding it into (*) to generate a number $N \in \mathbb{N}$. Associated with that N will be guarantees concerning the large- n behaviour of x_n that you can build with.
3. As a denier. To show that $x_n \not\rightarrow L$, you must prove “ $\neg(*)$ ”. Now $\neg(*)$ begins, “ $\exists \varepsilon > 0 \dots$ ”. (Practice: Write out the rest of this logical statement.) To prove a statement like this, it suffices to choose a particular $\varepsilon > 0$ and show that the outcome described in $\neg(*)$ follows. To prove that (x_n) diverges asks even more: one must show that $x_n \rightarrow L$ is false *for every choice of L* . In logic symbols, the statement to verify begins, “ $\forall L \in \mathbb{R}^d, \exists \varepsilon > 0 : \dots$ ”. A powerful alternative approach comes up in the next section.

E. Cauchy Sequences

Definition. A sequence (x_n) in $\mathbb{R}^d$ is called **Cauchy** iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, |x_m - x_n| < \varepsilon. \quad (\text{C1})$$

An equivalent condition is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall p \in \mathbb{N}, |x_n - x_{n+p}| < \varepsilon. \quad (\text{C2})$$

[Practice: prove that (C1) $\Leftrightarrow$ (C2).]

Intuitively, a Cauchy sequence is one in which, given any tolerance $\varepsilon > 0$, it is eventually the case that all separation distances between sequence elements are below that tolerance. It's critically important that this holds for sequence indices in the appropriate tail region, even if the indices are extremely far apart. In other words, the sequence indices m, n in (C1) or the values of n and p in (C2) must be completely independent.

Proposition. *Every convergent sequence in $\mathbb{R}^d$ is a Cauchy sequence.*

Proof. Fix any convergent sequence (x_n) ; let $L = \lim_{n \rightarrow \infty} x_n$. Estimate

$$|x_m - x_n| \leq |x_m - L| + |L - x_n|.$$

Given $\varepsilon > 0$, define $\varepsilon' = \varepsilon/2$ and use it in the definition of convergence: this gives some $N' \in \mathbb{N}$ such that all $k > N'$ make $|x_k - L| < \varepsilon'$. Put $N = N' + 1$ and use it once with $k = m$ and again with $k = n$ to extend the above inequality:

$$m, n \geq N \implies |x_m - x_n| \leq |x_m - L| + |L - x_n| < \varepsilon' + \varepsilon' = \varepsilon.$$

////

2026-10-14

Corollary. *Any $\mathbb{R}^d$ -valued sequence (x_n) that is not Cauchy must diverge.*

Proof. This is the contrapositive form of the statement above.

////

The Corollary above is often an effective way to prove divergence. A sequence (x_n) fails to be Cauchy when it violates condition (C2), i.e., when it behaves like this:

$$\exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \geq N, p \in \mathbb{N} : |x_n - x_{n+p}| \geq \varepsilon. \quad \neg(\text{C2})$$

Example. For each $n \in \mathbb{N}$, let $x_n = (-1)^n$. The sequence (x_n) diverges.

Proof. Note that $|x_n - x_{n+1}| = 2$ for all n . So condition $\neg(\text{C2})$ holds with $\varepsilon = 1$ (for example).

////

Example. For each $n \in \mathbb{N}$, let $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. The sequence (s_n) diverges.

Proof. For any $n, p \in \mathbb{N}$,

$$\begin{aligned} s_{n+p} - s_n &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+p}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \\ &\geq \frac{p}{n+p}. \end{aligned}$$

The right side can be made close to 1 by choosing a large value for p . So, with $\neg(\text{C2})$ in mind, consider $\varepsilon = \frac{1}{2}$. Given any $N \in \mathbb{N}$, choose $n = N$ and $p = n$ to get

$$s_{n+p} - s_n \geq \frac{n}{2n} = \frac{1}{2} = \varepsilon.$$

This establishes $\neg(\text{C2})$. ////

Lemma. *Every Cauchy sequence in $\mathbb{R}^d$ is bounded, i.e., if (x_n) is a Cauchy sequence then there exists $R > 0$ such that $|x_m| \leq R$ for all $m \in \mathbb{N}$.*

Proof. Let (x_n) be an arbitrary Cauchy sequence. Take $\varepsilon = 1$ and apply definition (C1) to get $N \in \mathbb{N}$ so large that

$$\forall m, n \geq N, \quad |x_m - x_n| < 1.$$

Now $|x_m| \leq |x_m - x_n| + |x_n|$. Taking $n = N$, we deduce that $|x_m| \leq 1 + |x_N|$ for all $m \geq N$. This controls the magnitudes of all but a finite number of sequence entries. Treat those explicitly to complete the proof:

$$\forall m \in \mathbb{N}, \quad |x_m| \leq \max\{|x_1|, \dots, |x_{N-1}|, 1 + |x_N|\}.$$

(The number on the right side will serve as the required constant R .) ////

The next result is super important. The order-completeness of $\mathbb{R}$ makes it work.

Theorem. *For any sequence (x_n) in $\mathbb{R}^d$, TFAE:*

- (a) *the sequence (x_n) is Cauchy,*
- (b) *the sequence (x_n) converges.*

Proof. (b $\Rightarrow$ a) Shown previously.

(a $\Rightarrow$ b) It suffices to treat the case $d = 1$, because a sequence in $\mathbb{R}^d$ is Cauchy if and only if each of its component sequences is Cauchy. (Exercise.) So suppose (x_n) is a sequence of real numbers. For each $n \in \mathbb{N}$, define the tail set

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

According to the Lemma above, the given Cauchy sequence must be bounded. That is, some closed interval $[-R, R]$ contains every sequence entry. Therefore $T_n \subseteq [-R, R]$ for each n , so the next steps define sequences with values in $[-R, R]$:

$$i_n = \inf T_n = \inf_{k \geq n} x_k, \quad s_n = \sup T_n = \sup_{k \geq n} x_k, \quad n \in \mathbb{N}.$$

Observe that for each n , $T_n \supseteq T_{n+1}$, so

$$i_n \leq i_{n+1}, \quad s_n \geq s_{n+1}.$$

Therefore the sequences (i_n) and (s_n) are monotonic, and we have $i_n \rightarrow \alpha$ and $s_n \rightarrow \beta$ for the numbers

$$\alpha = \sup_{n \in \mathbb{N}} i_n = \sup \{i_n : n \in \mathbb{N}\}, \quad \beta = \inf_{n \in \mathbb{N}} s_n = \inf \{s_n : n \in \mathbb{N}\}.$$

For any pair $m, n \in \mathbb{N}$, pick any integer $N > \max\{m, n\}$ and use the obvious inequality $i_N \leq s_N$ to get

$$i_m \leq i_{m+1} \leq \cdots \leq i_N \leq s_N \leq \cdots \leq s_{n+1} \leq s_n.$$

This establishes the inequality $i_m \leq s_n$ for all $m, n \in \mathbb{N}$. For each fixed n , this shows that s_n is an upper bound for the set $\{i_m : m \in \mathbb{N}\}$, so s_n must dominate the set's *least* upper bound, i.e., $s_n \geq \alpha$. Therefore α is a lower bound for the set of numbers whose *greatest* lower bound is β , and we have $\alpha \leq \beta$. We deduce that

$$\forall m, n \in \mathbb{N}, \quad i_m \leq \alpha \leq \beta \leq s_n.$$

Now we bring in the Cauchy property. Given any $\varepsilon > 0$, pick N so large that $|x_m - x_n| < \varepsilon$ whenever $m, n \geq N$. Fix any $n \geq N$ and rearrange this:

$$\forall m \geq n, \quad x_n - \varepsilon < x_m < x_n + \varepsilon,$$

which implies (since all values in T_n are in the sandwich above)

$$x_n - \varepsilon \leq i_n \leq \alpha \leq \beta \leq s_n \leq x_n + \varepsilon. \quad (\dagger)$$

In particular, $\beta - \alpha \leq (x_n + \varepsilon) - (x_n - \varepsilon) = 2\varepsilon$. Since this holds for arbitrary $\varepsilon > 0$, we have $\alpha = \beta$.

Now let's show that $x_n \rightarrow \alpha$. Given any $\varepsilon > 0$, choose N as in the previous paragraph, and proceed to recover line $(\dagger)$. Split the outer inequalities in $(\dagger)$ as

$$x_n - \alpha \leq \varepsilon, \quad \beta - x_n \leq \varepsilon.$$

Knowing $\alpha = \beta$ makes this pair of inequalities equivalent to $|x_n - \alpha| \leq \varepsilon$, and this holds for all $n \geq N$ for the N described above. The proof is complete*. ////

* Our definition of convergence calls for a strict inequality, but we have seen in homework that the formulation with a non-strict inequality is equivalent.

F. Subsequences

2026-10-16

Definition. (Subsequence) Let $x: \mathbb{N} \rightarrow X$ be a sequence with values in some set X . A sequence $y: \mathbb{N} \rightarrow X$ is a **subsequence of** (x_n) if $y(k) = x(n(k))$ for some increasing function (sequence) $n: \mathbb{N} \rightarrow \mathbb{N}$, i.e.,

$$y_k = x_{n_k}, \quad \text{where } 1 \leq n_1 < n_2 < n_3 < \dots$$

(Note that the choice $n_k = k$ is allowed, so every sequence is a subsequence of itself.)

Alternative. Every subsequence corresponds to an infinite subset S of $\mathbb{N}$. It's possible to imagine notation that avoids putting subscripts on subscripts by subscripting the arrow in " $n \rightarrow \infty$ " instead, so that we could write

$$x_{n_k} \rightarrow L \text{ as } k \rightarrow \infty \quad \Longleftrightarrow \quad x_n \rightarrow L \text{ as } n \xrightarrow{S} \infty.$$

For $L \in \mathbb{R}$, the notation on the right would mean

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N (n \in S), |x_n - L| < \varepsilon.$$

To get back to more standard notation, we would define indices $n_1, n_2, \dots$ by enumerating S in increasing order. This is a lovely idea, that captures the essence of subsequences, but let's stick closer to the notation standardized by Rudin for a little while longer.

Example. Subsequences of $x_n = n$ include

$$(1, 3, 5, 7, \dots), (2, 4, 6, 8, \dots), (2, 3, 5, 8, 13, 21, \dots), \text{ even } (x_n) \text{ itself,}$$

but **not** $(3, 1, 4, 1, 5, 9, \dots)$ [order permuted and elements re-used].

Proposition. Let $(x_n)_n$ be a sequence in $\mathbb{R}^d$, and let $L \in \mathbb{R}^d$. TFAE:

- (a) (x_n) converges to L .
- (b) Every subsequence of (x_n) converges to L .

Proof. (a $\Rightarrow$ b) Obvious.

(b $\Rightarrow$ a) [Contraposition] Every sequence is a subsequence of itself, so this is a tautology: if (x_n) does not converge to L , then the subsequence (x_n) does not converge to L . ////

Exercise. Extend the Proposition above to the case of real-valued sequences that approach one of the extended values $\pm\infty$.

Here is a result so important that it is named after its inventors.

Theorem (Bolzano-Weierstrass). *In $\mathbb{R}^d$, every bounded sequence has a convergent subsequence.*

Proof. Suppose first that $d = 1$, and a bounded sequence (x_n) is given. Pick an interval $I_0 = [a_0, b_0]$ that contains each element x_k . Let $m_0 = \frac{1}{2}(a_0 + b_0)$ be the midpoint of I_0 . At least one of the intervals $[a_0, m_0]$ or $[m_0, b_0]$ must contain the values x_n for infinitely many n . Call that subinterval I_1 ; name its endpoints $[a_1, b_1]$ and then let $m_1 = \frac{1}{2}(a_1 + b_1)$ denote the midpoint. At least one of the subintervals $[a_1, m_1]$ or $[m_1, b_1]$ must contain x_n for infinitely many n . Name that subinterval I_2 . Continue: the result is a nested sequence of closed intervals $I_0, I_1, I_2, \dots$, such that $I_n = [a_n, b_n]$ has length

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \dots = \frac{b_0 - a_0}{2^n}, \quad n \in \mathbb{N}. \quad (\dagger)$$

Pick n_1 such that $x_{n_1} \in I_1$. For each $k \geq 1$, pick n_{k+1} such that $n_{k+1} > n_k$ and $x_{n_{k+1}} \in I_{k+1}$. (This is possible because $x_r \in I_{k+1}$ holds for infinitely many indices r .) This defines an increasing sequence $n_1 < n_2 < \dots$ for which $x_{n_k} \in I_k$ holds for every k . That subsequence converges. To see why, note that $x_{n_{k+p}} \in I_k$ for every $p \in \mathbb{N}$, so (by $(\dagger)$)

$$|x_{n_k} - x_{n_{k+p}}| \leq \frac{b_0 - a_0}{2^k}, \quad \forall k \in \mathbb{N}, p \in \mathbb{N}.$$

This implies that $(x_{n_k})_{k \in \mathbb{N}}$ is Cauchy, so it must converge.

Now for a bounded sequence in $\mathbb{R}^d$, the result follows from d applications of the scalar case. First extract a subsequence along which the first component converges. Then select a sub-subsequence along which the second component sequence converges. (The first component is now a subsequence of the one originally chosen, which still converges, to the same limit, as noted above.) Iterate d times to get the result. ////

Remark. (Foreshadowing.) It's essential that d is a fixed finite number in the iterative scheme outlined above. A more sophisticated argument (diagonalization) is needed if each x_n in some given sequence (x_n) is itself a sequence.

G. Lim Sup and Lim Inf

Extended Values. We have agreed to allow the extended values $\pm\infty$ (which are not real numbers) for both the *values* of $\inf(S)$ and $\sup(S)$ and the *elements* of the sets S . (See Rudin 1.23, p. 11.)

Definition. Let (x_n) be a real sequence. We define the **limit superior** (or **upper limit**) and **limit inferior** (or **lower limit**) as

$$\limsup_{n \rightarrow \infty} x_n = \inf_{n \in \mathbb{N}} \left[\sup_{k \geq n} x_k \right], \quad \liminf_{n \rightarrow \infty} x_n = \sup_{n \in \mathbb{N}} \left[\inf_{k \geq n} x_k \right].$$

To expand the notation above, define for each n the tail sets and extended values

$$\begin{aligned} T_n &= \{x_n, x_{n+1}, \dots\}, \\ s_n &= \sup \{x_k : k \geq n\} = \sup \{x_n, x_{n+1}, \dots\}, \\ i_n &= \inf \{x_k : k \geq n\} = \inf \{x_n, x_{n+1}, \dots\}. \end{aligned}$$

The definitions mean

$$\limsup_{n \rightarrow \infty} x_n = \inf \{s_n : n \in \mathbb{N}\}, \quad \liminf_{n \rightarrow \infty} x_n = \sup \{i_n : n \in \mathbb{N}\}.$$

Both quantities exist, with values in $\mathbb{R} \cup \{\pm\infty\}$, for absolutely any given sequence.

Discussion. With the notation above, we always have

$$i_n \leq i_{n+1}, \quad i_n \leq s_n, \quad s_{n+1} \leq s_n.$$

So the sequence of extended values (i_n) is nondecreasing, and the monotone convergence property in Section C shows that it approaches its supremum: in the extended-valued interpretation

$$\lim_{n \rightarrow \infty} i_n = \sup \{i_n : n \in \mathbb{N}\} = \liminf_{n \rightarrow \infty} x_n.$$

Similarly, the sequence (s_n) is nonincreasing, and we have

$$\lim_{n \rightarrow \infty} s_n = \inf \{s_n : n \in \mathbb{N}\} = \limsup_{n \rightarrow \infty} x_n.$$

So another phrasing for “lim sup” could be “the limit of the tail sups”. In particular, although our definitions make this non-obvious, the values of $\liminf x_n$ and $\limsup x_n$ are unaffected by changing any finite number of terms in the given sequence (x_n) : they depend exclusively on the sequence’s “tail behaviour”.

Alternatives. Rudin gives different-looking but logically equivalent definitions of $\limsup$ and $\liminf$ using subsequences. Please read his presentation in items 3.5, 3.15–3.20. (Both perspectives are worth knowing. The approach chosen here prioritizes the least upper bound property of $\mathbb{R}$, which is a key learning outcome in our course.)

Example. Discuss in detail: (a) $x_n = 1/n$, (b) $x_n = (-1)^n + 1/n$, (c) $x_n = n$.

Given any term-by-term inequality between two sequences, it’s safe to “take the lim sup of both sides”: the inequality will be preserved. That’s the point of the next result.

Lemma. For any sequences (x_n) and (y_n) such that $x_n \leq y_n$ for all n , one has

$$\liminf_{n \rightarrow \infty} x_n \leq \liminf_{n \rightarrow \infty} y_n \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} y_n.$$

Proof. For each $n \in \mathbb{N}$, define

$$\begin{aligned} T_n(x) &= \{x_n, x_{n+1}, x_{n+2}, \dots\}, \\ i_n(x) &= \inf T_n(x) = \inf \{x_n, x_{n+1}, x_{n+2}, \dots\}, \\ s_n(x) &= \sup T_n(x) = \sup \{x_n, x_{n+1}, x_{n+2}, \dots\}, \end{aligned}$$

with similar notation for $T_n(y)$, $i_n(y)$, $s_n(y)$. Observe that for each n , one has $i_n(x) \leq i_n(y)$. [Reason: Being a lower bound makes $i_n(x) \leq t$ for each $t \in T_n(x)$. The relationship between the given sequences makes $i_n(x) \leq t'$ for each $t' \in T_n(y)$. Among lower bounds for $T_n(y)$, the greatest is $i_n(y)$, so $i_n(x) \leq i_n(y)$.] Inserting the second inequality below gives

$$\forall n \in \mathbb{N}, \quad i_n(x) \leq i_n(y) \leq \sup_p i_p(y) = \sup_p \left[\inf_{k \geq p} y_k \right] = \liminf_{p \rightarrow \infty} y_p.$$

Thus the quantity on the right dominates each element of the set $\{i_n(x) : n \in \mathbb{N}\}$. Since the supremum is the *least* upper bound,

$$\liminf_{n \rightarrow \infty} x_n = \sup_n i_n(x) \leq \liminf_{p \rightarrow \infty} y_p.$$

A very similar line of reasoning, starting with $s_n(x) \leq s_n(y)$ for each n , leads to the analogous conclusion

$$\limsup_{n \rightarrow \infty} x_n = \inf_n s_n(x) \leq \inf_n s_n(y) = \limsup_{n \rightarrow \infty} y_n. \quad \text{////}$$

Caution. Limiting operations respect non-strict inequalities, but not strict ones. Look at $y_n = 1/n$ and $x_n = 0$. Clearly $y_n > x_n$ for each n , yet $\liminf_n y_n = 0 = \liminf_n x_n$. Of course $y_n > x_n$ implies $y_n \geq x_n$, so the Lemma does have some relevance in this situation, but the best it can give you is a non-strict inequality in the limit.

The English phrases “lower limit” and “upper limit” are reasonable synonyms for “lim inf” and “lim sup”, as shown next. Moreover, these complementary concepts completely capture our ordinary (and extended) concepts of limits.

Proposition. *Let (x_n) be a sequence with values in $\mathbb{R}$. Then*

- (a) $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$, and
- (b) for any $L \in \mathbb{R} \cup \{\pm\infty\}$, one has $x_n \rightarrow L$ if and only if $\limsup_{n \rightarrow \infty} x_n = L = \liminf_{n \rightarrow \infty} x_n$.

Proof. Recall the usual notation: for each n , let

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}, \quad i_n = \inf T_n = \inf_{k \geq n} x_k, \quad s_n = \sup T_n = \sup_{k \geq n} x_k.$$

Let $\alpha = \sup_n i_n = \liminf x_n$ and $\beta = \inf_n s_n = \limsup x_n$.

- (a) Our proof that every Cauchy sequence converges, in Section E, established $\alpha \leq \beta$ before invoking the Cauchy condition. Although it was presented in the case where boundedness guarantees a finite value for both α and β , the same writeup, word for word, sponsors the same conclusion when one or both of these quantities is $-\infty$ or $+\infty$.
- (b) ($\Rightarrow$) Suppose $x_n \rightarrow L$, with $L \in \mathbb{R} \cup \{+\infty\}$. For any real $\gamma < L$, the definition provides some $N \in \mathbb{N}$ such that

$$\gamma < x_n, \quad \forall n > N.$$

That is, γ is a lower bound for the set T_{N+1} , and this forces

$$\gamma \leq i_{N+1} \leq \sup_n i_n = \liminf_{n \rightarrow \infty} x_n.$$

Since this holds for all $\gamma < L$, we have $L \leq \liminf_{n \rightarrow \infty} x_n$; in conjunction with part (a), we have

$$L \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n, \quad \text{for } L \in \mathbb{R} \cup \{+\infty\}. \quad (\dagger)$$

Similarly, suppose $x_n \rightarrow L$ with $L \in \mathbb{R} \cup \{-\infty\}$. For any real $\lambda > L$, the definition provides some $N \in \mathbb{N}$ such that

$$x_n < \lambda, \quad \forall n > N.$$

This implies that λ is an upper bound for the set T_{N+1} , and therefore

$$\lambda \geq s_{N+1} \geq \inf_n s_n = \limsup_{n \rightarrow \infty} x_n.$$

Since this holds for all $\lambda > L$, we must have $L \geq \limsup_{n \rightarrow \infty} x_n$. Recalling part (a), we have

$$L \geq \limsup_{n \rightarrow \infty} x_n \geq \liminf_{n \rightarrow \infty} x_n, \quad \text{for } L \in \mathbb{R} \cup \{-\infty\}. \quad (\ddagger)$$

Now consider the possibilities: if $L \in \mathbb{R}$, then both $(\dagger)$ and $(\ddagger)$ apply and the equations in (b) follow. If $L = -\infty$, then line $(\ddagger)$ is enough to confirm (b), and if $L = +\infty$, then (b) follows from $(\dagger)$.

($\Leftarrow$) Consider first the case where $L \in \mathbb{R} \cup \{+\infty\}$. Since $L = \sup_n i_n$, for any $\gamma < L$ there must exist some N_1 such that $i_{N_1} > \gamma$. In short

$$L > -\infty \implies \forall \gamma < L, \exists N_1 \in \mathbb{N} : \forall n \geq N_1, \quad x_n > \gamma. \quad (\#)$$

On the other hand, suppose $L \in \mathbb{R} \cup \{-\infty\}$. Since $L = \inf_n s_n$, for any $\lambda > L$ there must exist some N_2 such that $s_{N_2} < \lambda$. Put succinctly,

$$L < +\infty \implies \forall \lambda > L, \exists N_2 \in \mathbb{N} : \forall n \geq N_2, \quad x_n < \lambda. \quad (**)$$

Now consider the possibilities: If $L = +\infty$, then line (#) shows that $x_n \rightarrow +\infty$, and if $L = -\infty$ then line (**) shows that $x_n \rightarrow -\infty$. If $L \in \mathbb{R}$, then both lines apply. Given any $\varepsilon > 0$, choose $\gamma = L - \varepsilon$ in (#) to get an integer N_1 and choose $\lambda = L + \varepsilon$ in (**) to get an integer N_2 , and chain together the guaranteed inequalities:

$$\forall n > N \stackrel{\text{def}}{=} \max\{N_1, N_2\}, \quad L - \varepsilon < x_n < L + \varepsilon.$$

This confirms the definition of $x_n \rightarrow L$. ////

Proposition. Let (x_n) be a real sequence; define $\mu = \liminf_{n \rightarrow \infty} x_n$, $M = \limsup_{n \rightarrow \infty} x_n$.

- (a) If $\ell = \lim_k x_{n_k}$ for some subsequence $(x_{n_k})_k$, then $\mu \leq \ell \leq M$.
 - (b) There exist subsequences $(x_{n_j})_j$ and $(x_{n_k})_k$ obeying $x_{n_j} \rightarrow \mu$ and $x_{n_k} \rightarrow M$.
- (Note: values $\pm\infty$ allowed for μ, ℓ, M throughout.)

Proof. For each $n \in \mathbb{N}$, define

$$T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}, \quad i_n = \inf T_n = \inf_{k \geq n} x_k, \quad s_n = \sup T_n = \sup_{k \geq n} x_k.$$

- (a) For each $k \in \mathbb{N}$, $x_{n_k} \geq \inf\{x_j : j \geq n_k\} = i_{n_k}$. As $k \rightarrow \infty$, we have $x_{n_k} \rightarrow \ell$ and $i_{n_k} \rightarrow \mu$: hence $\ell \geq \mu$. Similarly, $x_{n_k} \leq s_{n_k} \forall k$, so $\ell \leq M$.
- (b) Let's build $(x_{n_k})_k$ with $x_{n_k} \xrightarrow[k]{>} M$. (Similar arguments work for μ .)

Case 1: $M = +\infty$. Here $\inf_n s_n = +\infty$, so $s_n = +\infty$ for each n . This means each tail T_n has no upper bound; we can use this property repeatedly to build the required subsequence.

Since T_1 has no upper bound, there exists $n_1 \geq 1$ for which $x_{n_1} > 1$. Consider T_{n_1} : this set has no upper bound, so there exists $n_2 > n_1$ such that $x_{n_2} > \max\{2, x_{n_1}\}$. Continue. After choosing $n_1, \dots, n_k$, note that $s_{n_k} = +\infty$, so there must exist some $n_{k+1} > n_k$ such that $x_{n_{k+1}} > \max\{k+1, x_{n_k}\}$. By induction, this defines an increasing subsequence along which $x_{n_k} \geq k$ for each $k \in \mathbb{N}$. Clearly $x_{n_k} \rightarrow +\infty$ as $k \rightarrow \infty$.

Case 2: $M \in \mathbb{R}$. Pick $n_1 = 1$. For each $k \geq 2$, define $r_k = M - 1/k$ and $R_k = M + 1/k$ and apply this two-step reasoning:

- (i) Since $R_k > M = \inf_n s_n$, some N_k must obey $R_k > s_{N_k}$. In particular, $R_k > x_n$ for all $n \geq N_k$. Define $\hat{N}_k = 1 + \max\{N_k, n_1, \dots, n_{k-1}\}$.
- (ii) Since $r_k < M = \inf_n s_n$, we certainly have $r_k < s_{1+\hat{N}_k}$. Hence there exists $n_k > \hat{N}_k$ such that $x_{n_k} > r_k$. Since $n_k > N_k$ also, it is eligible for the inequality in (i). Thus

$$M - \frac{1}{k} = r_k < x_{n_k} < R_k = M + \frac{1}{k}.$$

This construction works for all k : it guarantees both $n_k < n_{k+1}$ for all k and (by the Squeeze Theorem) $x_{n_k} \rightarrow M$ as $k \rightarrow \infty$.

Case 3: $M = -\infty$. Exercise. [We always have $\mu \leq M$. So $M = -\infty$ forces $\mu = -\infty$, i.e., $M = \mu$. This is equivalent to $x_n \rightarrow -\infty$ by part (b) in the Proposition above. The original sequence is a subsequence of itself.] /////

H. Proof Tips

Many homework problems request a proof that $x_n \rightarrow L$ for some given sequence (x_n) and extended real value L . This amounts to proving the inequalities

$$L \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq L. \quad (\text{LL})$$

(The middle inequality always holds; the outer ones might be “interesting”.)

Let’s discuss the case of (LL) in which L is a real number. One way to prove it starts by wrapping the number L in an arbitrary open interval (α, β) . That is, we invent symbols α and β for which $\alpha < L < \beta$ is all we know. We treat these new elements as constants at first. That gives us access to positive numbers $\varepsilon_1 = L - \alpha$ and $\varepsilon_2 = \beta - L$ that can provide the margins or tolerances that are so typical in our definitions. We put these to work, trying to establish the following relaxed version of the chain of inequalities in (LL):

$$\alpha \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq \beta. \quad (\dagger)$$

If we can get $(\dagger)$ for arbitrary choices of α and β as above, then we can easily harvest (LL). Just imagine inventing sequences $\alpha_k = L - 1/k$ and $\beta_k = L + 1/k$ and using those in $(\dagger)$ to get

$$\forall k \in \mathbb{N}, \quad L - \frac{1}{k} \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq L + \frac{1}{k}.$$

In the limit as $k \rightarrow \infty$, the numbers sandwiched in the middle don’t depend on k , and the other (non-strict) inequalities are respected by the limiting operation, so (LL) is the result. (This is essentially the Squeeze Theorem discussed previously.)

Having seen this idea once, it gets tedious to repeat the story about the sequences α_k and β_k every single time. We can just show the inequalities in $(\dagger)$ and say, “Since this holds for arbitrary $\alpha < L$ and $\beta > L$, we can now let $\alpha \rightarrow L^-$ and $\beta \rightarrow L^+$ to obtain (LL).”

The approach above admits natural adaptations to handle proofs in which the desired values are $L = -\infty$ or $L = +\infty$, or in which there is only one inequality to prove.