UBC Mathematics 320(101)—Assignment 2
Due by PDF upload to Canvas at 18:00, Saturday 23 Sep 2023

Readings: Loewen, lecture notes for Week 2; Rudin, pages 24–30.

1. The *Fibonacci sequence* is defined recursively by saying $F(0) = 1$, $F(1) = 1$, and

$$F(n) = F(n-1) + F(n-2) \text{ for each } n = 2, 3, \ldots.$$

Prove that if $\varphi = \dfrac{1+\sqrt{5}}{2}$ denotes the "Golden Ratio" (observe $\varphi^2 = \varphi + 1$), we have

$$F(n) = \frac{\varphi^{n+1} - (-\varphi)^{-n-1}}{\sqrt{5}}, \qquad n = 1, 2, 3, \ldots.$$

2. Construct a countable family of increasing sequences with entries from $\mathbb{N}$:

$$s^{(1)} = \left(s_1^{(1)}, s_2^{(1)}, s_3^{(1)}, \ldots\right), \; s^{(2)} = \left(s_1^{(2)}, s_2^{(2)}, s_3^{(2)}, \ldots\right), \; s^{(3)} = \left(s_1^{(3)}, s_2^{(3)}, s_3^{(3)}, \ldots\right), \; \ldots.$$

Arrange your construction so that every positive integer appears in one and only one of the sequences, and the inequality $s_n^{(i)} < s_{n+1}^{(i)}$ holds for all $i, n \in \mathbb{N}$. Illustrate your construction by writing down explicitly the first four entries of sequences $s^{(1)}$, $s^{(2)}$, $s^{(3)}$, and $s^{(4)}$.

3. Let $\mathcal{S}$ denote the collection (set) of $\mathbb{N}$-valued sequences that are increasing. That is, each object s in $\mathcal{S}$ has the form

$$s = (s_1, s_2, s_3, \ldots), \quad \text{where} \quad \forall k \in \mathbb{N},\; s_k < s_{k+1}.$$

Decide if the set $\mathcal{S}$ is countable or uncountable. Prove your answer.

4. Prove that the real intervals $I = [0, 1]$ and $J = (0, 1)$ have the same cardinal number by constructing an explicit bijection $\phi \colon I \to J$.

5. Taking as given an enumeration of the rationals as

$$\mathbb{Q} = \{q_1, q_2, q_3, \ldots\},$$

construct an explicit bijection f from $\mathbb{R}$ to $\mathbb{R} \setminus \mathbb{Q}$. To confirm that your bijection is explicit enough, return decimal approximations (correct to 6 significant digits) for these four numbers: $f(\pi)$, $f(\sqrt{3})$, $f(q_2)$, and $f(q_3)$. (*Hint*: It is well known that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. You don't need to prove this.)

6. (a) Show that the set of polynomials with integer coefficients is countable.

(b) Show that the set of real numbers x arising as zeros of polynomials with integer coefficients is countable. (Such a real number is called *algebraic*.)

(c) A real number that is not algebraic is called *transcendental.*
Prove that the set of transcendental numbers is not empty.

(d) Is the set of transcendental numbers finite, countable, or uncountable? Why?

7. Extend the definition of "+" to cardinal numbers, as follows. Given cardinals α and β, let $\alpha+\beta = |A \cup B|$, where A and B are disjoint sets such that $|A| = \alpha$ and $|B| = \beta$.

(a) Prove that this extension is well-defined. That is, show that if $|A| = |C|$ and $|B| = |D|$, with $A \cap B = \emptyset$ and $C \cap D = \emptyset$, then $|A| + |B| = |C| + |D|$.

(b) Suppose $A \neq \emptyset$ and $n \in \mathbb{N}$. Show how to construct a continuum of mutually disjoint sets A_t, $t \in \mathbb{R}$, such that $|A_t| = |A|$ for each t. ("Mutually disjoint" means $A_s \cap A_t = \emptyset$ whenever $s \neq t$.)

(c) Show that $n + \aleph_0 = \aleph_0$ for any finite cardinal n.

(d) Show that $\aleph_0 + \aleph_0 = \aleph_0$.

(e) Show that $\aleph_0 + c = c$.

(f) Show that $c + c = c$.

8. For each family of real intervals $\mathcal{A}$ shown below, find $\bigcap \mathcal{A}$ and $\bigcup \mathcal{A}$:

(a) $\mathcal{A} = \left\{ \left[\frac{1}{n}, 1 - \frac{1}{n} \right] \;:\; n \in \mathbb{N},\; n \geq 2 \right\}$,

(b) $\mathcal{A} = \left\{ \left(-1 - \frac{1}{n}, 1 + \frac{1}{n} \right) \;:\; n \in \mathbb{N} \right\}$.

[Presentation: For each subproblem, define a set S and verify that it equals the given combination by proving two inclusions. You may assume the Archimedean Property.]