UBC Mathematics 320(101)—Assignment 1
Due by PDF upload to Canvas no later than 03:00, Monday 15 Sep 2025

1. Prove or disprove: For each $n \in \mathbb{N}$, $n^2 - n + 41$ is prime.

2. If A, B, and C are sets, prove that
(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(b) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$,
(c) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.

3. Let $f\colon A \to B$. Let C, C_1, and C_2 be subsets of A, and let D be a subset of B. Prove:
(a) If f is one-to-one, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.
(b) If f is 1-1, then $f^{-1}(f(C)) = C$.
(c) If f is onto, then $f\left(f^{-1}(D)\right) = D$.
In each part, find an inclusion relation (either "$\subseteq$" or "$\supseteq$") that can be used to replace the symbol "=" and produce a true statement even without the given hypothesis.
[Recall that $f^{-1}(y) = \{x \in A : f(x) = y\}$ is, in general, a set-valued operation. It is not safe to infer that f is invertible just because the symbol f^{-1} appears.]

4. For Question 3(a), construct a specific example in which the indicated equation fails. (Of course the given hypothesis will have to be false too.)
Repeat for parts 3(b) and 3(c).

5. Prove that there is no (a, b) in $\mathbb{Z} \times \mathbb{Z}$ for which $a^2 = 4b + 3$.
(*Hint*: Every integer a must be either even or odd.)

6. Let $f\colon A \to B$ and $g\colon B \to C$ be given functions. Use the symbol $g \circ f$ to denote the function from A to C defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$. Prove:
(a) If f and g are one-to-one, then $g \circ f$ is one-to-one.
(b) If $g \circ f$ is one-to-one, then f is one-to-one.
(c) If f is onto and $g \circ f$ is one-to-one, then g is one-to-one.
(d) It can happen that $g \circ f$ is one-to-one, but g is not. (To "prove" this, simply provide a specific example with the indicated properties.)

7. Working in $\mathbb{R}$, consider $\alpha \overset{\text{def}}{=} \sqrt{2}/2$. Prove that for every rational number p/q in $(0, 1)$ [written in lowest terms],

$$\alpha \notin \left(\frac{p}{q} - \frac{1}{4q^2}, \frac{p}{q} + \frac{1}{4q^2} \right). \qquad (*)$$

Deduce that α is not rational. Is α algebraic? Transcendental?
[*Hint*: It is well known that $q^2 - 2p^2 \neq 0$ for all positive integers p and q.]

8. Let $f\colon A \to B$ and let $C \subseteq A$.
(a) Proof or counterexample: $f(A \setminus C) \subseteq f(A) \setminus f(C)$.
(b) Proof or counterexample: $f(A \setminus C) \supseteq f(A) \setminus f(C)$.
(c) What condition on f will guarantee $f(A \setminus C) = f(A) \setminus f(C)$?
(Choose between "f is 1-1" and "f is onto"; prove that your answer is correct.)