VIII. Fields of Extremals

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In this section we restrict our attention to the case $n = 1$.

Definition. Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ be open and simply connected (no holes). Let $L \in C^2(\Omega \times \mathbb{R})$. To say that Ω is *covered by a field of extremals $\mathcal{F}$ with slope function ϕ* means that there is a C^1 function $\phi \colon \Omega \to \mathbb{R}$ such that for each (τ, ξ) in Ω, the solution x for the initial-value problem

$$\dot{x}(t) = \phi(t, x(t)), \qquad x(\tau) = \xi \tag{2}$$

is an extremal for L on an open interval including τ. The symbol $\mathcal{F}$ denotes the set of these solution curves.

In specific problems, a set of extremals indexed by one real parameter often defines a family $\mathcal{F}$ from which one can extract a suitable slope function ϕ. The general solution of (DEL) involves two parameters, so there are several ways to produce fields of extremals for any given Lagrangian.

Example. For $L(t, x, v) = \sqrt{1 + v^2}$, every straight line $x(t) = mt + b$ is an extremal.

(i) Choosing $m = 1$ produces the family $\mathcal{F}_1 = \{x(t) := t + b \,:\, b \in \mathbb{R}\}$. This covers $\Omega_1 = \mathbb{R}^2$, with slope function $\phi_1(t, x) = 1$.

(ii) Choosing $b = 0$ produces the family $\mathcal{F}_2 = \{x(t) := mt \,:\, m \in \mathbb{R}\}$. This covers $\Omega_2 = \{(t, x) \,:\, t > 0,\ x \in \mathbb{R}\}$, with slope function $\phi_2(t, x) = x/t$.

2026-03-30

Vector Calculus Review. Recall Green's Theorem, stated using variables (t, x):

$$\oint_{\mathcal{C}} P\,dt + Q\,dx = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial x} \right) dA(t, x).$$

Here $P = P(t, x)$ and $Q = Q(t, x)$ are given smooth functions, $\mathcal{C}$ is a closed loop in the (t, x)-plane traced counterclockwise, and $\mathcal{D}$ is the plane area inside $\mathcal{C}$. A useful consequence arises when we have two curves in the (t, x)-plane, say γ_1 and γ_2, that have the same starting and ending points. Then we can make a loop $\mathcal{C}$ by moving first along γ_1 and then coming back to the initial point along γ_2. If the functions P and Q happen to satisfy the identity

$$\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial x} = 0 \tag{1}$$

in the region between γ_1 and γ_2, we infer

$$\int_{\gamma_1} P\,dt + Q\,dx = \int_{\gamma_2} P\,dt + Q\,dx.$$

In short, the identity (1) implies path-independence for line integrals in the region where it holds.

These basic vector-calculus facts are relevant in the CoV.

Hilbert's Invariant Integral. Suppose an open set Ω is covered by a field of extremals with slope function ϕ. At every point (τ, ξ) in Ω, the identity in (2) provides an arc x along which

$$\frac{d}{dt} L_v(t, x(t), \phi(t, x(t))) = L_x(t, x(t), \phi(t, x(t))).$$

Expanding the left side and using $\dot{x}(t) = \phi(t, x(t))$ leads to

$$L_{vt} + L_{vx}\phi + L_{vv}(\phi_t + \phi_x\phi) = L_x. \qquad (*)$$

Here the evaluation points for each function are as before, but they can be specialized to $(\tau, \xi, \phi(\tau, \xi))$: thus $(*)$ becomes an identity valid at every point of Ω.

Now consider the functions

$$\begin{aligned} P(t,x) &= L(t, x, \phi(t,x)) - L_v(t, x, \phi(t,x))\phi(t,x), \\ Q(t,x) &= L_v(t, x, \phi(t,x)). \end{aligned}$$

Calculating with the Chain Rule gives

$$\begin{aligned} \frac{\partial Q}{\partial t} &= L_{vt} + L_{vv}\phi_t \\ \frac{\partial P}{\partial x} &= L_x + L_v\phi_x - (L_{vx} + L_{vv}\phi_x)\,\phi - L_v\phi_x = L_x - L_{vx}\phi - L_{vv}\phi_x\phi. \end{aligned}$$

Using $(*)$, we investigate the expression foreshadowed in (1):

$$\begin{aligned} \frac{\partial Q}{\partial t} - \frac{\partial P}{\partial x} &= L_{vt} + L_{vv}\phi_t - L_x + L_{vx}\phi + L_{vv}\phi_x\phi \\ &= L_{vt} + L_{vx}\phi + L_{vv}(\phi_t + \phi_x\phi) - L_x \\ &= 0. \end{aligned}$$

This is an identity valid at all points of Ω. As discussed above, it follows that line integrals in Ω are path-independent. In our application, the paths of particular interest arise as graphs of smooth arcs x: for any $x \in C^1[a,b]$, the natural parametrization $t \mapsto (t, x(t))$ gives $dx = \dot{x}(t)\,dt$ and motivates the definition

$$\begin{aligned} U[x] &= \int_a^b \left[P(t, x(t)) + Q(t, x(t))\dot{x}(t)\right]\,dt \\ &= \int_a^b \left[L(t, x(t), \phi(t, x(t))) - (\phi(t, x(t)) - \dot{x}(t))L_v(t, x(t), \phi(t, x(t)))\right]\,dt. \end{aligned}$$

Path independence gives the functional U its name: it is called *Hilbert's Invariant Integral.* (It also accounts for the symbol: in the original German, "invariant integral" is "Unabhaengigkeitsintegral".) What is more, if x is an extremal belonging to the family $\mathcal{F}$, then $\dot{x}(t) = \phi(t, x(t))$, so

$$U[x] = \int_a^b L(t, x(t), \dot{x}(t))\,dt = \Lambda[x] \qquad (3)$$

coincides with the objective value of x relative to L.

The following result refers to the Weierstrass Excess Function

$$\mathcal{E}(t, x, v, w) := L(t, x, w) - L(t, x, v) - L_v(t, x, v)(w - v). \qquad (4)$$

Recall the connection between $\mathcal{E}$ and the subgradient inequality: if (t, x) is a point where the single-variable function $w \mapsto L(t, x, w)$ is convex on $\mathbb{R}$, then one has $\mathcal{E}(t, x, v, w) \geq 0$ for all v and all w. (This is guaranteed when $L_{vv}(t, x, v) \geq 0$ for all v.)

Theorem (Sufficiency). *Let Ω be a simply connected open subset of $\mathbb{R}\times\mathbb{R}$ covered by a field of extremals $\mathcal{F}$ with slope function ϕ. For any extremal $\widehat{x}$ in $\mathcal{F}$ with domain $[a,b]$, and any arc x in Ω with the same endpoints as $\widehat{x}$, one has*

$$\Lambda[x] - \Lambda[\widehat{x}] = \int_a^b \mathcal{E}(t, x(t), \phi(t, x(t)), \dot{x}(t))\, dt.$$

In particular, if $\mathcal{E}(t,x,v,w) \geq 0$ for all (t,x,v,w) in $\Omega\times\mathbb{R}\times\mathbb{R}$, then every segment of every extremal in $\mathcal{F}$ is optimal relative to its endpoints.

Proof. Pick any extremal $\widehat{x}$ in $\mathcal{F}$ and any subinterval $[a,b]$ of its domain. For any arc x with the same endpoints as $\widehat{x}$ whose graph lies entirely inside Ω, exploit the path-independence property of U by using the add-subtract trick to calculate

$$\begin{aligned}
\Lambda[x] - \Lambda[\widehat{x}] &= (\Lambda[x] - U[x]) - (\Lambda[\widehat{x}] - U[\widehat{x}]) \quad (\text{since } U[x] = U[\widehat{x}]) \\
&= \Lambda[x] - U[x] \qquad (\text{by } (3)) \\
&= \int_a^b \left[L(t,x,\dot{x}) - L(t,x,\phi(t,x)) + (\phi(t,x) - \dot{x})L_v(t,x,\phi(t,x))\right] dt \\
&= \int_a^b \mathcal{E}(t, x(t), \phi(t, x(t)), \dot{x}(t))\, dt.
\end{aligned}$$

When the integrand on the right is nonnegative everywhere, this implies $\Lambda[x] \geq \Lambda[\widehat{x}]$. ////

2026-03-30

Problem-Solving. To certify the optimality of a candidate extremal $\widehat{x}$ using the theorem above, it suffices to identify a domain Ω covered by a field of extremals $\mathcal{F}$ such that $\widehat{x} \in \mathcal{F}$. In standard terminology, we "embed" $\widehat{x}$ in a field of extremals.

Example. Consider the integrand $L(t,x,v) = v^2/t$. Provided $t > 0$, this function is strictly convex in v, so we will have $\mathcal{E}(t,x,v,w) \geq 0$ for arbitrary (t,x,v,w). Further, any extremal for L will be a C^2 solution of

$$\frac{d}{dt}\left(\frac{2\dot{x}}{t}\right) = 0, \text{ i.e., } \frac{2\dot{x}}{t} = 4\alpha, \text{ i.e., } x(t) = \alpha t^2 + \beta,\ \alpha,\ \beta \in \mathbb{R}.$$

To be specific, imagine the Basic Problem with endpoints $(1,1)$ and $(3,9)$. The unique admissible extremal is $\widehat{x}(t) = t^2$. Various families of parabolas include this arc.

- The arc $\widehat{x}$ can be embedded in the family of translated parabolas $\mathcal{F}_1 = \left\{x(t) := t^2 + \beta \,:\, \beta \in \mathbb{R},\ t > 0\right\}$, whose graphs fill the half-space $\Omega = \{(t,x) \,:\, t > 0\}$. To check the required properties, pick any (τ,ξ) in Ω and require $x(\tau) = \xi$: this leads to the unique choice $\beta = \xi - \tau^2$. So

$$x(t) = t^2 + \beta = t^2 - \tau^2 + \xi, \quad \text{giving} \quad \dot{x}(t) = 2t, \quad \text{and} \quad \dot{x}(\tau) = 2\tau.$$

Therefore the definition $\phi_1(\tau,\xi) = 2\tau$ provides a slope function compatible with every extremal in the family $\mathcal{F}_1$. This reasoning applies at every point of Ω, establishing that $\phi_1(t,x) = 2t$ is a slope function compatible with the selected field of extremals. According to the theorem above, the arc $\widehat{x}$ (and indeed every other arc in $\mathcal{F}_1$ with graph in Ω) is globally optimal relative to its endpoints.

• Alternatively, consider the family $\mathcal{F}_2 = \left\{x(t) = \alpha t^2 : \alpha \in \mathbb{R},\ t > 0\right\}$. These also cover $\Omega = \{(t,x) : t > 0\}$. The unique parabola through (τ,ξ) is determined by selecting α to make $\xi = \alpha\tau^2$, leading to

$$x(t) = \frac{\xi}{\tau^2} t^2.$$

This choice reveals $\phi(\tau,\xi) = \dot{x}(\tau) = 2\xi/\tau$. Since this reasoning applies at every point of Ω, the slope function $\phi_2(t,x) = 2x/t$ is compatible with the field $\mathcal{F}_2$. Thus the theorem above certifies the optimality of $\widehat{x}$, and indeed of every other arc in the family $\mathcal{F}_2$.

• A third field that includes $\widehat{x}$ is $\mathcal{F}_3 = \left\{x(t) := \alpha t^2 + \alpha - 1 : \alpha \in \mathbb{R},\ t > 0\right\}$. Here the condition $x(\tau) = \xi$ requires

$$\xi = \alpha\tau^2 + \alpha - 1, \text{ i.e., } \alpha = \frac{\xi + 1}{\tau^2 + 1}.$$

This determines a unique extremal from the family:

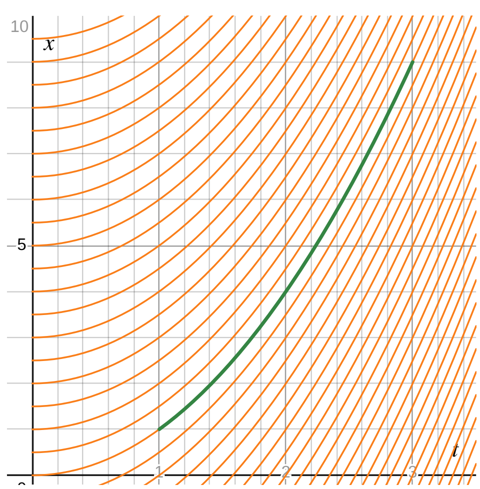
$$x(t) = \frac{\xi + 1}{\tau^2 + 1} t^2 + \frac{\xi + 1}{\tau^2 + 1} - 1.$$

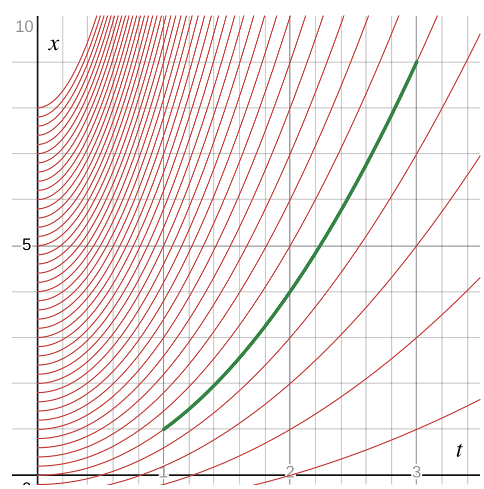
The slope of this extremal at the point (τ,ξ) is

$$\dot{x}(\tau) = \left[\frac{\xi + 1}{\tau^2 + 1}(2t)\right]_{t=\tau} = \frac{2(\xi + 1)\tau}{\tau^2 + 1}.$$

Defining the slope function $\phi_3(t,x) = \dfrac{2(x+1)t}{t^2+1}$ completes the construction of a suitable embedding field. Then the theorem above verifies that each parabola in the family $\mathcal{F}_3$, including our chosen $\widehat{x}$, is optimal relative to its endpoints.

Here are sketches of the extremal families $\mathcal{F}_1$ and $\mathcal{F}_3$ defined above. The selected segment $\widehat{x}$ is highlighted.





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Theoretical Applications. One can prove a general sufficiency theorem by identifying abstract situations where that extremal can be embedded in a suitable field, so that the given extremal will be a local minimum of some kind. This is an alternative approach to the convexity-based method we used for the same purpose. The final conclusions are similar.

Value Functions. In the context of vector calculus, the path-independence condition emerging from (1) implies that the vector field $\langle P, Q\rangle$ must be the negative gradient of some "potential" W, and that the line integrals of $\langle P, Q\rangle$ must be easy to evaluate in terms of the difference in W-values at their endpoints. As noted above, if the selected endpoints happen to be linked by an extremal from the family $\mathcal{F}$, then the line integral will return the minimum integral cost of moving from one to the other as assessed by L. It's productive to think further about how the minimum integral cost depends on the endpoints in any given situation.

Example (ctd). In the examples involving $L(t,x,v) = v^2/t$ above, we have

$$L_v = \frac{2v}{t}, \quad \text{so} \quad L - L_v v = -\frac{v^2}{t}.$$

Thus the definitions of P and Q in the discussion take the specific forms

$$\begin{aligned} P(t,x) &= L(t,x,\phi(t,x)) - L_v(t,x,\phi(t,x))\phi(t,x) = -\frac{1}{t}\phi(t,x)^2, \\ Q(t,x) &= L_v(t,x,\phi(t,x)) = \frac{2}{t}\phi(t,x). \end{aligned}$$

Each choice of slope field ϕ leads to a different potential function. Here are some details. (For simplicity, we don't put subscripts on the functions P and Q. These are different in each derivation, but their roles are strictly internal to each brief derivation.)

- For $\phi_1(t,x) = 2t$, we have $P = -4t$ and $Q = 4$, and standard methods reveal the function $W_1(t,x) = 2t^2 - 4x$, for which
$$-\nabla W_1(t,x) = \langle -4t, 4\rangle = \langle P, Q\rangle\,.$$
- For $\phi_2(t,x) = \dfrac{2x}{t}$, we have $P = -\dfrac{4x^2}{t^3}$ and $Q = \dfrac{4x}{t^2}$, and standard methods reveal the function $W_2(t,x) = -\dfrac{2x^2}{t^2}$, for which
$$-\nabla W_2(t,x) = \left\langle -\frac{4x^2}{t^3}, \frac{4x}{t^2}\right\rangle = \langle P, Q\rangle\,.$$
- For $\phi_3(t,x) = \dfrac{2t(x+1)}{t^2+1}$, we have $P = -\dfrac{4t(x+1)^2}{(t^2+1)^2}$ and $Q = \dfrac{4(x+1)}{t^2+1}$, and standard methods reveal the function $W_3(t,x) = -\dfrac{2(x+1)^2}{t^2+1}$, for which
$$-\nabla W_3(t,x) = \left\langle -\frac{4t(x+1)^2}{(t^2+1)^2}, \frac{4(x+1)}{t^2+1}\right\rangle = \langle P, Q\rangle\,.$$

Later we will be interested in the Hamiltonian function $H(t,x,p) = \dfrac{1}{4}tp^2$. Direct substitution will confirm that each of the potential functions W_j shown above is a solution for the following partial differential equation in a suitable open region Ω:

$$W_t - H(t,x,-W_x) = 0, \qquad (t,x) \in \Omega.$$