## M402(201) Solutions—Assignment 1

UBC M402 Resources by Philip D. Loewen

**1.** Consider the following variational problem, for which the constant arc $x_0(t) = 1$ is admissible:

$$\min \left\{ \Lambda[x] \stackrel{\text{def}}{=} \int_1^3 t(\dot{x}^2(t) - x^2(t))\,dt \,:\, x(1) = 1,\ x(3) = 1 \right\}. \qquad (P)$$

Use whatever software you choose (including "none"), to help complete the activities below.

(a) One admissible variation is $h_1(t) = (t-1)(t-3)$. Find the [quadratic] function

$$\phi(\lambda) = \Lambda[x_0 + \lambda h_1]$$

and sketch its graph. Then find the $\lambda$-value that minimizes $\phi$ and the corresponding arc $x = x_0 + \lambda h_1$.

(b) Imagine using a variation built from two ingredients, each with its own scale factor. To be specific, keep $h_1$ from part (a), invent $h_2(t) = (t-1)(t-2)(t-3)$, and consider the 2-parameter family of admissible arcs

$$x(t; \lambda_1, \lambda_2) = x_0(t) + \lambda_1 h_1(t) + \lambda_2 h_2(t), \qquad 1 \le t \le 3.$$

Let

$$f(\lambda_1, \lambda_2) = \Lambda[x(\cdot; \lambda_1, \lambda_2)].$$

Write this [quadratic] function explicitly, and sketch its graph. Then find the point $(\lambda_1, \lambda_2)$ that minimizes $f$ and the corresponding arc $x$.

(c) On the same set of axes, sketch the reference arc and the improvements found in parts (a) and (b). Calculate and compare the $\Lambda$-values for these three arcs.

*Please note*: If you opt for software assistance, please ...

- Report all inexact (computed) values with five or more significant figures,
- Include enough computer output to enable someone of modest skills to reproduce your work,
- Organize your submission so the answers above are easy to find.

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For the constant reference arc $x_0(t) = 1$, the integral value is

$$\Lambda[x_0] = \int_1^3 t\left(0^2 - 1^2\right)\,dt = \left[-\frac{t^2}{2}\right]_{t=1}^3 = -4.$$

(a) For $x_0(t) = 1$ and $h_1(t) = (t-1)(t-3) = t^2 - 4t + 3$, we have $\dot{x}_0(t) = 0$ and $\dot{h}_1(t) = 2t - 4$, so

$$\phi(\lambda) = \Lambda[x_0 + \lambda h_1] = \int_1^3 t\left(\lambda^2\,[2t-4]^2 - \left[1 + \lambda(t^2 - 4t + 3)\right]^2\right)\,dt = A\lambda^2 + D\lambda + F,$$

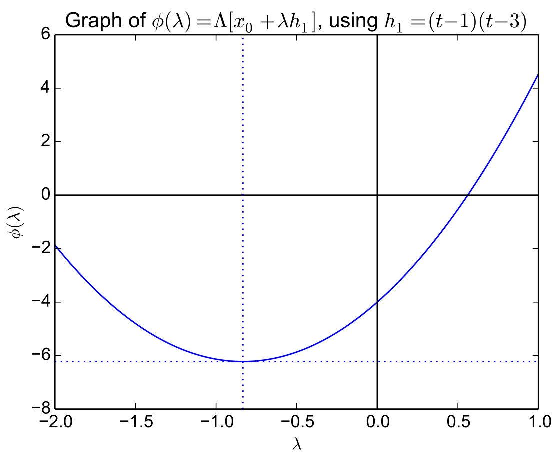
for constant coefficients given by

$$A = \int_1^3 t\left[7 + 8t - 18t^2 + 8t^3 - t^4\right]\,dt = \frac{16}{5},$$
$$D = \int_1^3 t\left[-6 + 8t - 2t^2\right]\,dt = \frac{16}{3},$$
$$F = \int_1^3 t\left[-1\right]\,dt = -4.$$

(Checking that $\phi(0) = \Lambda[x_0]$ is reassuring at this point.) The convex quadratic $\phi$, sketched below, takes its minimum value at the point where

$$0 = \phi'(\lambda) = 2A\lambda + B = \frac{16}{5}(2\lambda) + \frac{16}{3}, \quad \text{i.e.,} \quad \lambda = -\frac{5}{6}.$$

Here is a sketch. Note that $\phi(0) = \Lambda[x_0] = -4$:




The minimum value of $\phi$ is $\phi(-5/6) = -56/9 \approx -6.2222$: this is the integral value associated with the improved arc

$$x^{(a)}(t) = x_0(t) - \frac{5}{6}h_1(t) = 1 - \frac{5}{6}(t-1)(t-3).$$

(b) The given definitions produce the 2-parameter family of arcs

$$\begin{aligned} x(t;\lambda_1,\lambda_2) &= x_0(t) + \lambda_1 h_1(t) + \lambda_2 h_2(t) \\ &= 1 + \lambda_1(t-1)(t-3) + \lambda_2(t-1)(t-2)(t-3) \\ &= 1 + \lambda_1\left(t^2 - 4t + 3\right) + \lambda_2\left(t^3 - 6t^2 + 11t - 6\right), \qquad 1 \le t \le 3. \end{aligned}$$

The corresponding derivatives are simply

$$\dot{x}(t;\lambda_1,\lambda_2) = \lambda_1\left[2t-4\right] + \lambda_2\left[3t^2 - 12t + 11\right],$$

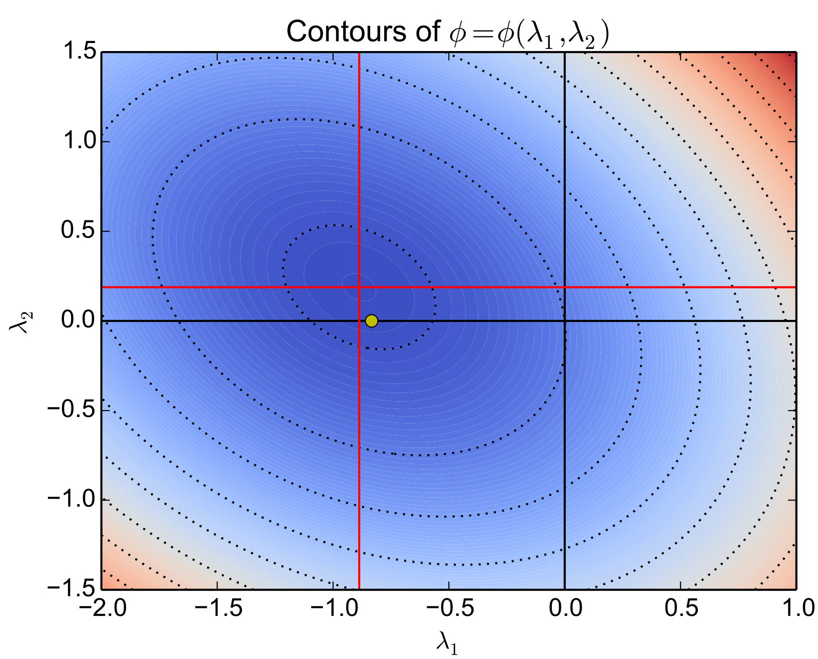
so $f(\lambda_1,\lambda_2) = \Lambda[x(\cdot;\lambda_1,\lambda_2)]$ will turn out to be a quadratic function of $(\lambda_1,\lambda_2)$. Grinding out the coefficients by hand is not a trivial matter, although the basic idea is simple. A fully symbolic approach is actually more manageable: for general functions $h_1$, $h_2$,

$$\begin{aligned}\Lambda[1+\lambda_1 h_1+\lambda_2 h_2] &= -4+\int_1^3 t\left[\begin{matrix}(\dot h_1^2-h_1^2)\lambda_1^2+(\dot h_2^2-h_2^2)\lambda_2^2+2\left(\dot h_1\dot h_2-h_1h_2\right)\lambda_1\lambda_2\\ -\,2h_1\lambda_1-2h_2\lambda_2\end{matrix}\right]dt\\ &= A\lambda_1^2+2B\lambda_1\lambda_2+C\lambda_2^2+D\lambda_1+E\lambda_2-4.\end{aligned}$$

For the suggested variations above,

$$\begin{aligned}A &= \int_1^3 t\left(\dot h_1^2-h_1^2\right)dt=\frac{16}{5}\\ B &= \int_1^3 t\left(\dot h_1\dot h_2-h_1h_2\right)dt=\frac{32}{35}\\ C &= \int_1^3 t\left(\dot h_2^2-h_2^2\right)dt=\frac{304}{105}\\ D &= -2\int_1^3 th_1(t)\,dt=\frac{16}{3}\\ E &= -2\int_1^3 th_2(t)\,dt=\frac{8}{15}\end{aligned}$$

(Note that setting $\lambda_2 = 0$ amounts to applying only the single variation $h_1$, so we have $f(\lambda_1, 0) = \phi(\lambda_1)$ for the function $\phi$ studied in part (a). Thus the coefficients $A$ and $D$ are the same here as in (a).) These numbers come from Maple. The graph of $f$ is a typical convex paraboloid. Here is a sketch of its contours in the region of interest:



The point $(\lambda_1, \lambda_2)$ that minimizes $f$ must be a critical point, i.e.,

$$0 = \frac{\partial f}{\partial \lambda_1} = 2A\lambda_1 + 2B\lambda_2 + D, \quad 0 = \frac{\partial f}{\partial \lambda_2} = 2B\lambda_1 + 2C\lambda_2 + E.$$

We know the values of $A$, $B$, $C$, $D$, $E$ here, so it is a routine matter to solve for

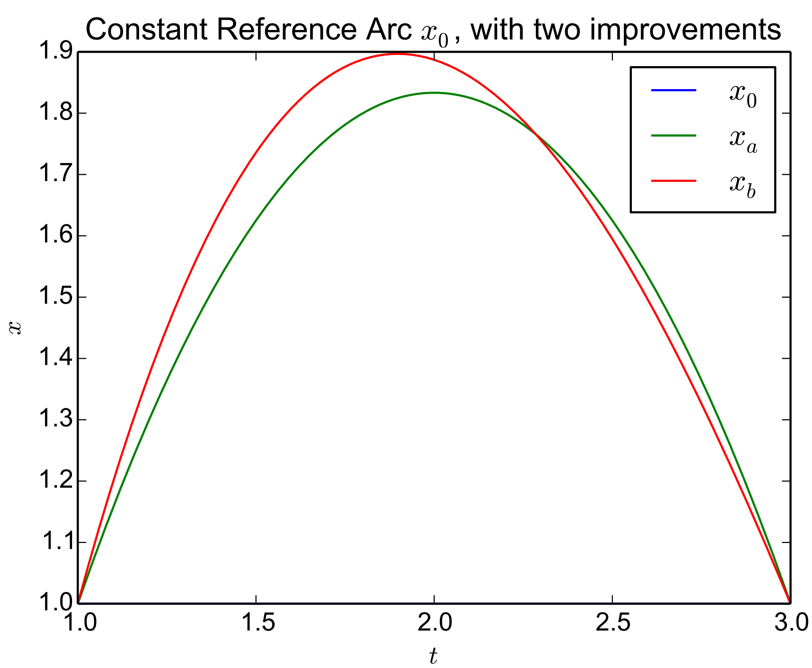
$$\lambda_1 = -\frac{322}{363} \approx -0.88705, \quad \lambda_2 = \frac{91}{484} \approx 0.18802, \qquad f(\lambda_1, \lambda_2) = -\frac{34387}{5445} \approx -6.3153.$$

*NOTE*: The one-variable problem in part (a) can be recovered by consistently choosing $\lambda_2 = 0$ here. Graphically, the problem in part (a) is to find the smallest possible value on the horizontal axis in the contour plot shown above. The solution of part (a) is highlighted as a yellow dot in the sketch. It's important to observe that the minimizing point over the whole $(\lambda_1, \lambda_2)$-plane cannot be located by choosing $\lambda_1 = \lambda_1^*$ to minimize $\phi(\lambda_1, 0)$ and then minimizing the one-variable function $\lambda_2 \mapsto \phi(\lambda_1^*, \lambda_2)$. (The minimizer in this second problem will lie on a vertical line through the yellow dot, and the point we seek is somewhere else.)

(c) In summary, we have

$$\begin{aligned}
\Lambda[x_0] &= -4 && \text{for } x_0(t) = 1,\\
\Lambda[x_a] &= -\frac{56}{9} \approx -6.2222 && \text{for } x_a(t) = 1 - \frac{5}{6}(t-1)(t-3),\\
\Lambda[x_b] &= -\frac{34387}{5445} \approx -6.3153 && \text{for } x_b(t) = 1 - \frac{322}{363}(t-1)(t-3) + \frac{91}{484}(t-1)(t-2)(t-3).
\end{aligned}$$

The three arcs detailed here are shown in the following sketch:



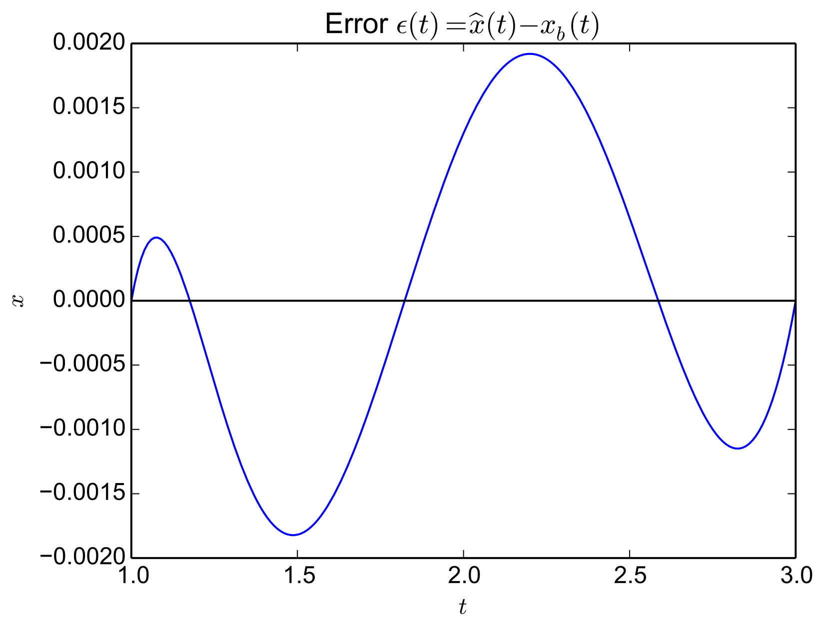

**Discussion (Not Required for Credit).** For $L(t, x, v) = tv^2 - tx^2$, every extremal must be $C^2$ and the Euler-Lagrange equation for an unknown arc $x$ is

$$\frac{d}{dt}\left[2t\dot{x}(t)\right] = -2tx(t), \quad \text{i.e.,} \quad t\ddot{x}(t) + \dot{x}(t) + tx(t) = 0.$$

This is Bessel's equation of order 0. Its general solution is $x(t) = c_1 J_0(t) + c_2 Y_0(t)$ for famous special functions $J_0$, $Y_0$. The choices $c_1 = 0.92701$ and $c_2 = 3.29327$ (approximately) satisfy the given endpoint conditions, so, "If the stated variational problem has a smooth solution, then that solution must be $\widehat{x}(t) \approx c_1 J_0(t) + c_2 Y_0(t)$ for the constants $c_1, c_2$ identified above." The integral value for $\widehat{x}$ is

$$\Lambda[\widehat{x}] \approx -6.31547.$$

It can be shown that this is (except for rounding errors) the true minimum value in the problem. It is very close to the value $\Lambda[x_b]$ associated with the cubic function calculated in part (b) above. In fact, the graph of $\widehat{x}$ would be indistinguishable from the graph of $x_b$ in the sketch provided in part (b). Here is a plot showing the (small) discrepancies between the true minimizer $\widehat{x}$ and the approximate solution $x_b$:



**2.** Consider this general Lagrangian involving continuously differentiable coefficients $m, q, k, f, g, u$:

$$L(t,x,v) = \tfrac{1}{2}m(t)v^2 + q(t)xv - \tfrac{1}{2}k(t)x^2 + f(t)x + g(t)v + u(t).$$

For a given interval $[a,b]$, use this $L$ to define the functional

$$\Lambda[x(\cdot)] = \int_a^b L(t,x(t),\dot{x}(t))\,dt.$$

(a) Suppose a smooth function $x_0\colon [a,b] \to \mathbb{R}$ is given ("the reference arc") together with some smooth $h\colon [a,b] \to \mathbb{R}$ satisfying $h(a) = 0 = h(b)$ ("a variation"). Write integral expressions independent of $\lambda$ for $B$ and $C$ in the identity

$$\Lambda[x_0 + \lambda h] = \Lambda[x_0] + 2\lambda B + C\lambda^2.$$

(b) With $x_0$ and $h$ as described in part (a), determine the function $R$ (depending on $x_0$, but independent of $h$) for which

$$\Lambda'[x_0; h] = \lim_{\lambda\to 0} \frac{\Lambda[x_0 + \lambda h] - \Lambda[x_0]}{\lambda} = \int_a^b R(t)h(t)\,dt.$$

(c) The assertion that "$R(t) = 0$ for each $t$ in $[a,b]$" is, by definition, the Euler-Lagrange equation for the reference arc $x_0$. Notice that certain changes to $L$ make no difference to the Euler-Lagrange equation, e.g.,

(i) replacing the coefficient function $u$ with 0, or

(ii) replacing the coefficient pair $(f,g)$ with the pair $(f - \dot{g}, 0)$.

Explain both of these observations by describing how the proposed changes influence the values of the original functional $\Lambda$.

(d) Prove: If $m(t) > 0$, $q(t) = q_0$ is constant, and $k(t) < 0$ for all $t$ in $[a,b]$, then any reference arc $x_0$ satisfying the Euler-Lagrange equation provides a **unique global minimizer** for $\Lambda$ among all competing arcs $x$ with the same endpoints (i.e., competitors must have $x(a) = x_0(a)$ and $x(b) = x_0(b)$).

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(a) In writing the functional

$$\Lambda[x(\cdot)] = \int_a^b \left[\tfrac{1}{2}m(t)\dot{x}^2 + q(t)x\dot{x} - \tfrac{1}{2}k(t)x^2 + f(t)x + g(t)\dot{x} + u(t)\right] dt,$$

we can save some writing by abbreviating $m(t)$ as $m$, etc. Then for any arcs $x$ and $y$,

$$\begin{aligned}
\Lambda[x+y] - \Lambda[x] &= \int_a^b \left[\tfrac{1}{2}m(\dot{x}+\dot{y})^2 + q(x+y)(\dot{x}+\dot{y}) - \tfrac{1}{2}k(x+y)^2 + f(x+y) + g(\dot{x}+\dot{y}) + u\right] dt \\
&\quad - \int_a^b \left[\tfrac{1}{2}m\dot{x}^2 + qx\dot{x} - \tfrac{1}{2}kx^2 + fx + g\dot{x} + u\right] dt \\
&= \int_a^b \left[\tfrac{1}{2}m(2\dot{x}\dot{y} + \dot{y}^2) + q(x\dot{y} + y\dot{x} + y\dot{y}) - \tfrac{1}{2}k(2xy + y^2) + fy + g\dot{y}\right] dt \\
&= \int_a^b [m\dot{x}\dot{y} + q(x\dot{y} + y\dot{x}) - kxy + fy + g\dot{y}]\,dt + \int_a^b \left[\tfrac{1}{2}m\dot{y}^2 + qy\dot{y} - \tfrac{1}{2}ky^2\right] dt.
\end{aligned}$$

Each term in the first integral contains one factor of either $y$ or $\dot{y}$, and each term in the second integral contains two. Thus, substituting $x = x_0$ and $y = \lambda h$ produces an expression of the form

$$\Lambda[x_0 + \lambda h] - \Lambda[x_0] = 2\lambda B + \lambda^2 C$$

where

$$B = \frac{1}{2}\int_a^b \left[m\dot{x}_0\dot{h} + q(x_0\dot{h} + h\dot{x}_0) - kx_0h + fh + g\dot{h}\right] dt,$$
$$C = \int_a^b \left[\tfrac{1}{2}m\dot{h}^2 + qh\dot{h} - \tfrac{1}{2}kh^2\right] dt.$$

(b) Now with the notation in part (a),

$$\begin{aligned}\Lambda'[x_0; h] &= \lim_{\lambda\to 0} \frac{\Lambda[x_0 + \lambda h] - \Lambda[x_0]}{\lambda}\\ &= \lim_{\lambda\to 0}(2B + \lambda C) = 2B = \int_a^b \left[m\dot{x}_0\dot{h} + q(x_0\dot{h} + h\dot{x}_0) - kx_0h + fh + g\dot{h}\right] dt\\ &= \int_a^b \left[(m\dot{x}_0 + qx_0 + g)\,\dot{h} + (q\dot{x}_0 - kx_0 + f)\,h\right] dt.\end{aligned}$$

To arrange the requested form, integrate by parts to get

$$\int_a^b (m\dot{x}_0 + qx_0 + g)\,\dot{h}\,dt = (m\dot{x}_0 + qx_0 + g)\,h\Big|_{t=a}^{b} - \int_a^b h\frac{d}{dt}(m\dot{x}_0 + qx_0 + g)\,dt.$$

Now the conditions $h(a) = 0 = h(b)$ imply that the integrated term is 0. So using this result in the expression above leads to

$$\Lambda'[x_0; h] = \int_a^b \left[(q\dot{x}_0 - kx_0 + f) - \frac{d}{dt}(m\dot{x}_0 + qx_0 + g)\right] h(t)\,dt = \int_a^b R(t)h(t)\,dt,$$

where

$$R(t) = (q\dot{x}_0 - kx_0 + f) - \frac{d}{dt}(m\dot{x}_0 + qx_0 + g).$$

(c) (i) The function $R$ found in (b) has no dependence at all on the given function $u$. This makes sense because the role of $u$ in defining $\Lambda[x]$ is only to add the constant

$$U \stackrel{\text{def}}{=} \int_a^b u(t)\,dt.$$

Just as in ordinary calculus, adding a constant to a given function makes no difference to that function's derivative, or to the location of its critical points. (Of course the critical *values* are affected, but that's a separate consideration.)

(ii) Rearrangement shows

$$R(t) = (f - \dot{g}) + (q\dot{x}_0 - kx_0) - \frac{d}{dt}(m\dot{x}_0 + qx_0).$$

The functions $f$ and $g$ appear only in the combination $(f - \dot{g})$, which is unchanged if we replace the pair $(f, g)$ with $(f - \dot{g}, 0)$. Back in the original definition of $\Lambda$, the difference between using these pairs is

$$\int_a^b (fx + g\dot{x})\, dt - \int_a^b \left( (f - \dot{g})\, x + 0\dot{x} \right)\, dt = \int_a^b (g\dot{x}\dot{g}x)\, dt$$
$$= g(t)x(t)\big|_{x=a}^{b} = g(b)x(b) - g(a)x(a).$$

As in part (i), this difference is a constant independent of the input arc $x()$, as long as we focus on arcs for which the endpoint values $x(a)$ and $x(b)$ are given. Therefore the Euler-Lagrange equation (which concerns *derivatives* of $\Lambda$) is insensitive to this change.

(d) Suppose $x_0$ is an arc for which the Euler-Lagrange equation holds. In the notation of part (a), this means that for any variation $h$ with $h(a) = 0 = h(b)$, we have $B = 0$ and therefore (with $\lambda = 1$)

$$\Lambda[x_0 + h] = \Lambda[x_0] + C = \Lambda[x_0] + \int_a^b \left[ \tfrac{1}{2} m\dot{h}^2 + qh\dot{h} - \tfrac{1}{2} kh^2 \right] dt.$$

Now if $q(t) = q_0$ is constant, then

$$\int_a^b qh\dot{h}\, dt = q_0 \int_a^b \frac{d}{dt} \left( \tfrac{1}{2} h(t)^2 \right) dt = q_0 \left[ \frac{h(t)^2}{2} \right]_{t=a}^{b} = 0,$$

so the term involving $q$ above evaluates to 0, leaving

$$\Lambda[x_0 + h] = \Lambda[x_0] + \int_a^b \left[ \tfrac{1}{2} m(t)\dot{h}^2 + \tfrac{1}{2}(-k(t))h^2 \right] dt.$$

Knowing both $m(t) > 0$ and $k(t) < 0$ for all $t$ leads to the conclusion that

$$\Lambda[x_0 + h] \geq \Lambda[x_0] + 0,$$

with a strict inequality in all cases where the variation $h()$ is not the constant function 0. In particular, if $x$ is any arc with the same endpoints as $x_0$, so $x(a) = x_0(a)$ and $x(b) = x_0(b)$, then defining $h = x - x_0$ produces an arc for which $h(a) = 0 = h(b)$, so we have

$$\Lambda[x] = \Lambda[x_0 + h] \geq \Lambda[x_0],$$

with equality if and only if $x - x_0$ is the constant function 0. In other words, $x_0$ provides a unique global minimum for $\Lambda$ among all arcs with the same endpoints.

**3.** For each Lagrangian below, write the Euler-Lagrange equation and find all $C^2$ solutions.

(a) $L(t,x,v) = v^2 - \alpha^2 x^2$, $\alpha > 0$,

(b) $L(t,x,v) = v^2 + \alpha^2 x^2$, $\alpha > 0$,

(c) $L(t,x,v) = v^2 + x^2 - 2(\sin t)x$,

(d) $L(t,x,v) = v^2 - 6t^2 x$,

(e) $L(t,x,v) = (v-x)^2 + 2e^t x$.

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**(a)** For $L(t,x,v) = v^2 - \alpha^2 x^2$, $\alpha > 0$, one has $L_v = 2v$ and $L_x = -2\alpha^2 x$.

$$\text{(DEL)} \iff \frac{d}{dt}(2\dot{x}(t)) = -2\alpha^2 x(t) \iff \ddot{x}(t) + \alpha^2 x(t) = 0.$$

This has general solution $x(t) = A\cos\alpha t + B\sin\alpha t$, $A, B \in \mathbb{R}$.

**(b)** For $L(t,x,v) = v^2 + \alpha^2 x^2$, $\alpha > 0$, one has $L_v = 2v$ and $L_x = 2\alpha^2 x$.

$$\text{(DEL)} \iff \frac{d}{dt}(2\dot{x}(t)) = 2\alpha^2 x(t) \iff \ddot{x}(t) - \alpha^2 x(t) = 0.$$

This has general solution $x(t) = Ae^{\alpha t} + Be^{-\alpha t}$, $A, B \in \mathbb{R}$; an equivalent form is $x(t) = C\cosh(\alpha t) + D\sinh(\alpha t)$, $C, D \in \mathbb{R}$.

**(c)** For $L(t,x,v) = v^2 + x^2 - 2(\sin t)x$, one has $L_v = 2v$ and $L_x = 2x - 2\sin t$.

$$\text{(DEL)} \iff \frac{d}{dt}(2\dot{x}(t)) = 2x(t) - 2\sin t \iff \ddot{x}(t) - x(t) = -\sin t.$$

The homogeneous equation $\ddot{x} - x = 0$ has the general solution given in (b), with $\alpha = 1$. To find a particular solution, guess $x_p(t) = c\cos t + d\sin t$ and plug in:

$$[-c\cos t - d\sin t] - [c\cos t + d\sin t] = -\sin t.$$

This equation holds for $c = 0$, $d = \frac{1}{2}$, so $x_p(t) = \frac{1}{2}\sin t$ and the desired general solution is

$$x(t) = A\cosh t + B\sinh t + \tfrac{1}{2}\sin t, \ \ A, B \in \mathbb{R}.$$

**(d)** For $L(t,x,v) = v^2 - 6t^2 x$, one has $L_v = 2v$ and $L_x = -6t^2$.

$$\text{(DEL)} \iff \frac{d}{dt}(2\dot{x}(t)) = -6t^2 \iff \ddot{x}(t) = -3t^2.$$

The desired general solution is $x(t) = -\frac{1}{4}t^4 + At + B$, $A, B \in \mathbb{R}$.

**(e)** For $L(t,x,v) = (v-x)^2 + 2e^t x$, one has $L_v = 2(v-x)$ and $L_x = -2(v-x) + 2e^t$.

$$\text{(DEL)} \iff \frac{d}{dt}(2\dot{x}(t) - 2x(t)) = -2(\dot{x}(t) - x(t)) + 2e^t \iff \ddot{x}(t) - x(t) = e^t.$$

The homogeneous equation $\ddot{x} - x = 0$ has general solution given in (b) with $\alpha = 1$. This time the right-hand side is already a solution of the homogeneous equation, so a particular solution must have the form $x_p(t) = kte^t$ for some $k$. Plug this in to get $2ke^t = e^t$, so $k = \frac{1}{2}$ and the desired general solution is

$$x(t) = A\cosh t + B\sinh t + \tfrac{1}{2}te^t, \ \ A, B \in \mathbb{R}.$$

(An equivalent description of the same set of functions is $x(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2}te^t$, $c_1, c_2 \in \mathbb{R}$.)

**Discussion.** Searching for solutions $\widehat{x}$ of (IEL) in the larger class of $C^1$ functions generates no new arcs in cases (a)–(e) above. To explain this, consider any $C^1$ extremal $\widehat{x}$. Then, in each part of this question, $\widehat{L}_v(t) = 2\dot{\widehat{x}}(t) - 2c\widehat{x}(t)$ for some constant $c$ (with $c = 0$ in (a)–(d), $c = 1$ in (e)). Rearranging $\dot{\widehat{x}}(t) = \frac{1}{2}\widehat{L}_v(t) + c\widehat{x}(t)$ expresses $\dot{\widehat{x}}$ as a sum of $C^1$ functions; consequently $\widehat{x} \in C^2$.

In fact, even enlarging the competition further to allow piecewise smooth solutions of (IEL) generates no new extremals. To see why, recall that any extremal must satisfy condition (WE1), i.e., $\widehat{L}_v(t^-) = \widehat{L}_v(t^+)$, at all times interior to the interval on which the problem is posed. In all cases above (using $\widehat{x}(t^-) = \widehat{x}(t^+)$ for part (e)), this condition reduces to

$$\dot{\widehat{x}}(t^-) = \dot{\widehat{x}}(t^+) \qquad \forall t. \tag{$\dagger$}$$

This implies that corner points are impossible for solutions of (IEL), so every extremal is $C^1$, and the reasoning in the previous paragraph applies.