UBC Mathematics 402(101)—Assignment 2
Due by Canvas upload by 18:00 PT on Saturday 23 Sep 2023

1. (a) Find the unique admissible extremal $\widehat{x}$ in the problem

$$\min\left\{\int_1^2 \left[2x^2(t)+t^2\dot{x}^2(t)\right]\,dt \;:\; x(1)=1,\; x(2)=5\right\}.$$

(Hint: The Euler equation has two solutions of the form t^p.)

(b) Prove that $\widehat{x}$ is the global solution to this problem.

2. Consider the following functional with a quadratic integrand:

$$\Lambda[y]=\int_0^{3\pi/2}\left(\dot{y}(t)^2-y(t)^2\right)\,dt.$$

(a) Find an arc h with $h(0)=0$, $h(3\pi/2)=0$, and $\Lambda[h]<0$.

(b) Find all extremals (if any) for Λ compatible with the endpoint conditions $y(0)=0$, $y(3\pi/2)=0$.

(c) Display a sequence of arcs $y_k\in C^1[0,3\pi/2]$, each satisfying $y_k(0)=0$, $y_k(3\pi/2)=0$, such that

$$\Lambda[y_k]\to-\infty \quad\text{as}\quad k\to\infty.$$

(d) Continuing with Λ as given, consider the problem of minimizing $\Lambda[x]$ over all $x\in C^1[0,3\pi/2]$ subject to new endpoint conditions,

$$x(0)=0,\qquad x(3\pi/2)=B,$$

where B is some given constant. Find all admissible extremals, and then show that none of them gives even a directional local minimizer. That is, show that for any admissible extremal z there is an admissible variation h for which the 1-variable function

$$\phi(\lambda)=\Lambda[z+\lambda h]$$

does not have a local minimum at the point $\lambda=0$.

3. Given $k,\ell\colon\mathbb{R}\xrightarrow{C^1}\mathbb{R}$, define $\Phi\colon C^1[a,b]\to\mathbb{R}$ as follows:

$$\Phi[x]=k(x(a))+\ell(x(b)),\qquad x\in C^1[a,b].$$

(a) Show that for every $\widehat{x}$ in $C^1[a,b]$, the derivative operator $D\Phi[\widehat{x}]\colon C^1[a,b]\to\mathbb{R}$ is well-defined and linear.

(b) Use the abstract theory discussed in class to complete the following statement of first-order necessary conditions in terms of k and ℓ, and then to prove it:

If an arc $\widehat{x}\in C^1[a,b]$ gives a DLM for Φ, then ….

(c) Let $\Lambda[x] = \displaystyle\int_a^b L(t, x(t), \dot{x}(t))\, dt$, where $L \in C^1$. Suppose $\widehat{x} \in C^1[a,b]$ gives a DLM for $\Lambda + \Phi$. Prove that $\widehat{x}$ satisfies not only (IEL), but also the endpoint conditions

$$\widehat{L}_v(a) = k'(\widehat{x}(a)), \quad -\widehat{L}_v(b) = \ell'(\widehat{x}(b)).$$

4. Consider the following problem:

$$\min\left\{\int_0^1 \left[tx^2(t) + t^2 x(t)\right]\, dt \,:\, x \in PWS[0,1],\ x(0) = A,\ x(1) = B\right\}.$$

Show that a solution can only exist for certain values of A and B. Find these values, and the unique candidate for the minimizing arc. Then use your ingenuity to prove that this arc truly provides a *unique global minimum*.
[Clue: Call the arc you find $\widehat{x}$, and show that $\Lambda[x] - \Lambda[\widehat{x}] > 0$ is true for every arc $x \neq \widehat{x}$ satisfying the endpoint conditions.]