M402(101) Solutions—Assignment 2

UBC M402 Resources by Philip D. Loewen

1. (a) Find the unique admissible extremal $\widehat{x}$ in the problem

$$\min\left\{\int_1^2 \left[2x^2(t)+t^2\dot{x}^2(t)\right]\,dt \,:\, x(1)=1,\ x(2)=5\right\}.$$

(Hint: The Euler equation has two solutions of the form t^p.)

(b) Prove that $\widehat{x}$ is the global solution to this problem.

(a) Here $L = 2x^2 + t^2v^2$, so

$$L_x = 4x, \qquad L_v = 2t^2 v, \qquad L_{vv} = 2t^2.$$

Since $L \in C^2$ and $L_{vv} > 0$ for all $v \in \mathbb{R}$ and all (t,x) of interest, every extremal must be a C^2 solution of (DEL):

$$\frac{d}{dt}\left[2t^2\dot{x}(t)\right] = 4x(t), \qquad \text{i.e.,} \qquad t^2\ddot{x}(t) + 2t\dot{x}(t) - 2x(t) = 0.$$

This is a linear equation of "Euler type" (what a coincidence!), for which we expect solutions of the form $x(t) = t^p$. Substitution gives

$$0 = t^2\left[p(p-1)t^{p-2}\right] + 2\left[pt^{p-1}\right] - 2t^p = t^p\left[p^2+p-2\right] = t^p(p+2)(p-1).$$

So the general solution of (DEL) is

$$x(t) = At + \frac{B}{t^2}, \qquad A, B \in \mathbb{R},$$

and the initial conditions give $1 = x(1) = A + B$, $5 = x(2) = 2A + B/4$, with the unique solutions $A = 19/7$, $B = -12/7$. Hence the unique candidate for optimality is

$$\widehat{x}(t) = \frac{1}{7}\left[19t - \frac{12}{t^2}\right].$$

Note $\dot{\widehat{x}} = A - 2B/t^3$ and $\ddot{\widehat{x}} = 6B/t^4$.

(b) For any $h \in V_{II}$, expanding quadratics in the definition of Λ gives

$$\begin{aligned}\Lambda[\widehat{x}+h] &= \int_1^2 \left[2(\widehat{x}+h)^2 + t^2(\dot{\widehat{x}}+\dot{h})^2\right] \\ &= \Lambda[\widehat{x}] + \int_1^2 \left[4\widehat{x}h + 2t^2\dot{\widehat{x}}\dot{h}\right]\,dt + \Lambda[h].\end{aligned}$$

The middle integral vanishes, thanks to integration by parts and (DEL):

$$\begin{aligned} &\int_1^2 \left(2t^2\dot{\widehat{x}}(t)\right)\dot{h}(t)\,dt = 2t^2\dot{\widehat{x}}(t)h(t)\Big|_{t=1}^2 - \int_1^2 \frac{d}{dt}\left(2t^2\dot{\widehat{x}}\right)h(t)\,dt \\ \implies \quad &\int_1^2 \left[4\widehat{x}h + 2t^2\dot{\widehat{x}}\dot{h}\right]\,dt = \int_1^2 \left[4\widehat{x}(t) - \frac{d}{dt}\left(2t^2\dot{\widehat{x}}\right)\right]h(t)\,dt = 0.\end{aligned}$$

Thus we have $\Lambda[\widehat{x}+h] = \Lambda[\widehat{x}] + \Lambda[h]$ whenever $h \in V_{II}$, and clearly $\Lambda[h] \geq 0$ for all $h \in V_{II}$, so $\Lambda[\widehat{x}+h] \geq \Lambda[\widehat{x}]$.

2. Consider the following functional with a quadratic integrand:

$$\Lambda[y] = \int_0^{3\pi/2} \left(\dot{y}(t)^2 - y(t)^2\right)\, dt.$$

(a) Find an arc h with $h(0) = 0$, $h(3\pi/2) = 0$, and $\Lambda[h] < 0$.

(b) Find all extremals (if any) for Λ compatible with the endpoint conditions $y(0) = 0$, $y(3\pi/2) = 0$.

(c) Display a sequence of arcs $y_k \in C^1[0, 3\pi/2]$, each satisfying $y_k(0) = 0$, $y_k(3\pi/2) = 0$, such that

$$\Lambda[y_k] \to -\infty \quad \text{as} \quad k \to \infty.$$

(d) Continuing with Λ as given, consider the problem of minimizing $\Lambda[x]$ over all $x \in C^1[0, 3\pi/2]$ subject to new endpoint conditions,

$$x(0) = 0, \qquad x(3\pi/2) = B,$$

where B is some given constant. Find all admissible extremals, and then show that none of them gives even a directional local minimizer. That is, show that for any admissible extremal z there is an admissible variation h for which the 1-variable function

$$\phi(\lambda) = \Lambda[z + \lambda h]$$

does not have a local minimum at the point $\lambda = 0$.

(a) Many arcs $h \in V_{II}$ will make $\Lambda[h] < 0$. One possibility is $h(t) = \sin(2t/3)$, for which $\dot{h}(t) = (2/3)\cos(2t/3)$ and (using $\theta = 2t/3$)

$$\begin{aligned}
\Lambda[h] &= \int_{t=0}^{3\pi/2} \left[\frac{4}{9}\cos^2\left(\frac{2t}{3}\right) - \sin^2\left(\frac{2t}{3}\right)\right] dt \\
&= \int_{\theta=0}^{\pi} \left[\frac{4}{9}\cos^2\theta - \sin^2\theta\right] \frac{3}{2} d\theta \\
&= \frac{\pi}{2}\left[\frac{4}{9} - 1\right]\left(\frac{3}{2}\right) = -\frac{5\pi}{12}.
\end{aligned}$$

A second choice is the quadratic $h(t) = t(3\pi/2 - t) = \dfrac{3}{2}\pi t - t^2$, which has

$$\begin{aligned}
\Lambda[h] &= \int_{t=0}^{3\pi/2} \left[\left(\frac{3}{2}\pi - 2t\right)^2 - \left(\frac{3}{2}\pi t - t^2\right)^2\right] dt \\
&= \int_{t=0}^{3\pi/2} \left[\left(\frac{9}{4}\pi^2 - 6\pi t + 4t^2\right) - \left(\frac{9}{4}\pi^2 t^2 - 3\pi t^3 + t^4\right)\right] dt \\
&= \frac{9}{8}\pi^3 - \frac{81}{320}\pi^5 \approx -42.579.
\end{aligned}$$

(b) Here $L(t, x, v) = v^2 - x^2$ has only C^2 extremals, and each one of them must satisfy

$$\ddot{x}(t) + x(t) = 0.$$

The general solution is $x(t) = c_1 \cos(t) + c_2 \sin(t)$, $c_1, c_2 \in \mathbb{R}$. For admissibility, the constants must comply with the endpoint conditions

$$0 = x(0) = c_1, \qquad 0 = x(3\pi/2) = -c_2.$$

Thus there is exactly one admissible extremal, namely the constant function $\widehat{x}(t) = 0$.

(c) Take any particular arc $h \in V_{II}$ for which $\Lambda[h] < 0$ (part (a) offers two specific choices) and define $y_k(t) = kh(t)$. Then

$$\Lambda[y_k] = \int_{t=0}^{3\pi/2} \left(k^2 \dot{h}(t)^2 - k^2 h(t)^2\right) dt = k^2 \Lambda[h] \to -\infty \text{ as } k \to \infty.$$

(d) Every extremal x for Λ that starts from $x(0) = 0$ has the form $x(t) = c_2 \sin t$ for some c_2. The given endpoint condition identifies exactly one admissible extremal,

$$z(t) = -B \sin t.$$

Again taking the variation h constructed in part (a), we consider

$$\begin{aligned} \phi(\lambda) &= \Lambda[z + \lambda h] \\ &= \int_0^{3\pi/2} \left[\left(\dot{z}^2 + 2\lambda \dot{z}\dot{h} + \lambda^2 \dot{h}^2\right) - \left(z^2 + 2\lambda z h + \lambda^2 h^2\right) \right] dt \\ &= \Lambda[z] + 2\lambda \int_0^{3\pi/2} \left(\dot{z}\dot{h} - zh\right) dt + \lambda^2 \Lambda[h]. \end{aligned}$$

Now since $\ddot{z} = -z$, integration by parts shows that the middle term on the right is 0:

$$\int_0^{3\pi/2} \dot{z}\dot{h}\, dt = \dot{z}(t)h(t)\Big|_{t=0}^{3\pi/2} - \int_0^{3\pi/2} \ddot{z} h\, dt = \int_0^{3\pi/2} zh\, dt.$$

Since $\Lambda[h] < 0$, this implies

$$\phi(\lambda) = \Lambda[z] - |\Lambda[h]|\lambda^2.$$

So in the selected direction h, the functional Λ does not have a local minimum at z.

Discussion (not required for credit): Of course there are many (other) variations along which Λ does have a local minimum at z. A similar situation from the simpler world of finite-dimensional optimization occurs at a saddle point. At such a point, the directional derivative is 0 in every direction, but the function gets larger along some directions and smaller in others. Sometimes second-order analysis is enough to reveal this, and sometimes only higher-order approximations will detect it. For example, explore $f(x_1, x_2) = x_1^2 - x_2^2$ and $g(x_1, x_2) = x_1^4 - x_2^4$.

3. Given $k, \ell \colon \mathbb{R} \xrightarrow{C^1} \mathbb{R}$, define $\Phi \colon C^1[a,b] \to \mathbb{R}$ as follows:

$$\Phi[x] = k(x(a)) + \ell(x(b)), \qquad x \in C^1[a,b].$$

(a) Show that for every $\widehat{x}$ in $C^1[a,b]$, the derivative operator $D\Phi[\widehat{x}] \colon C^1[a,b] \to \mathbb{R}$ is well-defined and linear.

(b) Use the abstract theory discussed in class to complete the following statement of first-order necessary conditions in terms of k and ℓ, and then to prove it:

If an arc $\widehat{x} \in C^1[a,b]$ gives a DLM for Φ, then ....

(c) Let $\Lambda[x] = \displaystyle\int_a^b L(t, x(t), \dot{x}(t))\, dt$, where $L \in C^1$. Suppose $\widehat{x} \in C^1[a,b]$ gives a DLM for $\Lambda + \Phi$. Prove that $\widehat{x}$ satisfies not only (IEL), but also the endpoint conditions

$$\widehat{L}_v(a) = k'(\widehat{x}(a)), \quad -\widehat{L}_v(b) = \ell'(\widehat{x}(b)).$$

(a) Fix $h \in C^1[a,b]$ and expand the definition:

$$\begin{aligned}
D\Phi[\widehat{x}](h) = \Phi'[\widehat{x}; h] &= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[\Phi[\widehat{x} + \lambda h] - \Phi[\widehat{x}]\right] \\
&= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[k(\widehat{x}(a) + \lambda h(a)) + \ell(\widehat{x}(b) + \lambda h(b)) - k(\widehat{x}(a)) - \ell(\widehat{x}(b))\right] \\
&\overset{(1)}{=} \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[k(\widehat{x}(a) + \lambda h(a)) - k(\widehat{x}(a))\right] \\
&\qquad\qquad + \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[\ell(\widehat{x}(b) + \lambda h(b)) - \ell(\widehat{x}(b))\right] \\
&\overset{(2)}{=} \frac{d}{d\lambda} \left[k(\widehat{x}(a) + \lambda h(a))\right]_{\lambda=0} + \frac{d}{d\lambda} \left[\ell(\widehat{x}(b) + \lambda h(b))\right]_{\lambda=0} \\
&\overset{(3)}{=} k'(\widehat{x}(a))h(a) + \ell'(\widehat{x}(b))h(b).
\end{aligned}$$

Here equations (2)–(3) follow from the chain rule, since $k, \ell \in C^1$. In particular, both limits on the right of (1) exist, so the limit on the left exists too, and equation (1) holds. Thus the calculation above is valid for all $h \in C^1[a,b]$, so the operator $D\Phi[\widehat{x}] \colon C^1[a,b] \to \mathbb{R}$ is well-defined. To prove linearity, we must confirm that

$$D\Phi[\widehat{x}](c_1 h_1 + c_2 h_2) = c_1 D\Phi[\widehat{x}](h_1) + c_2 D\Phi[\widehat{x}](h_2) \quad \forall c_1, c_2 \in \mathbb{R},\ \forall h_1, h_2 \in C^1[a,b].$$

This follows directly by substitution. (“It’s obvious.”)

(b) *If an arc $\widehat{x} \in C^1[a,b]$ gives a DLM for Φ, then $k'(\widehat{x}(a)) = 0$ and $\ell'(\widehat{x}(b)) = 0$.*

Proof. Our abstract theory says that for all $h \in C^1[a,b]$,

$$0 = k'(\widehat{x}(a))h(a) + \ell'(\widehat{x}(b))h(b).$$

Choose $h(t) = t - b$ (for example) to get $k'(\widehat{x}(a)) = 0$, and $h(t) = t - a$ (for example) to get $\ell'(\widehat{x}(b)) = 0$. ////

Discussion. The result is supposed to be obvious: although the inputs $x(a)$ and $x(b)$ to functions k and ℓ are supposed to be linked by a C^1 arc, there is such an enormous variety of arcs that the inputs are essentially independent of each other. The condition above is just the same as the one we would obtain by saying that the function of two variables $(r,s) \mapsto k(r)+\ell(s)$ has a DLM over $\mathbb{R}^2$ at the point $(\widehat{r},\widehat{s}) = (\widehat{x}(a),\widehat{x}(b))$.

(c) If $\widehat{x} \in C^1[a,b]$ gives a DLM for $\Lambda + \Phi$, our general theory says that for all $h \in C^1[a,b]$,

$$\begin{aligned} 0 &= D(\Lambda+\Phi)[\widehat{x}](h) = D\Lambda[\widehat{x}](h) + D\Phi[\widehat{x}](h) \\ &= k'(\widehat{x}(a))h(a) + \left(\int_a^b \widehat{L}_x(r)\,dr + \ell'(\widehat{x}(b))\right)h(b) \\ &\qquad + \int_a^b \left(\widehat{L}_v(t) - \int_a^t \widehat{L}_x(r)\,dr\right)\dot{h}(t)\,dt. \end{aligned} \tag{$*$}$$

If we look first in the subspace V_{II}, we find that

$$\int_a^b \left(\widehat{L}_v(t) - \int_a^t \widehat{L}_x(r)\,dr\right)\dot{h}(t)\,dt = 0 \qquad \forall h \in V_{II}.$$

By the DuBois-Reymond Lemma, this implies that some $c \in \mathbb{R}$ obeys

$$\widehat{L}_v(t) - \int_a^t \widehat{L}_x(r)\,dr = c, \qquad \forall t \in [a,b]. \tag{$**$}$$

In particular, we have $c = \widehat{L}_v(a)$ and $\displaystyle\int_a^b \widehat{L}_x(r)\,dr = \widehat{L}_v(b) - c$. This information about the arc $\widehat{x}$ can be used wherever $\widehat{x}$ is mentioned—in particular, in line $(*)$ above:

$$\begin{aligned} 0 &= k'(\widehat{x}(a))h(a) + \left(\widehat{L}_v(b) - c + \ell'(\widehat{x}(b))\right)h(b) + \int_a^b c\dot{h}(t)\,dt \\ &= \left[k'(\widehat{x}(a)) - c\right]h(a) + \left[\widehat{L}_v(b) + \ell'(\widehat{x}(b))\right]h(b), \qquad \forall h \in C^1[a,b]. \end{aligned}$$

Just as in part (b), the arbitrariness of h in this statement implies both

$$k'(\widehat{x}(a)) = c = \widehat{L}_v(a) \qquad \text{and} \qquad \widehat{L}_v(b) = -\ell'(\widehat{x}(b)).$$

4. Consider the following problem:

$$\min\left\{\int_0^1 \left[tx^2(t)+t^2x(t)\right]\,dt\;:\;x\in PWS[0,1],\ x(0)=A,\ x(1)=B\right\}.$$

Show that a solution can only exist for certain values of A and B. Find these values, and the unique candidate for the minimizing arc. Then use your ingenuity to prove that this arc truly provides a *unique global minimum*.
[Clue: Call the arc you find $\widehat{x}$, and show that $\Lambda[x]-\Lambda[\widehat{x}]>0$ is true for every arc $x\neq\widehat{x}$ satisfying the endpoint conditions.]

Here the integrand is $L(t,x,v)=tx^2+t^2x$, a function independent of v. Thus the Integral Euler-Lagrange equation for an unknown arc $\widehat{x}$ reduces to

$$0=c+\int_0^t \widehat{L}_x(r)\,dr=c+\int_0^t\left[2t\widehat{x}(t)+t^2\right]\,dt\qquad\forall t\in(0,1).$$

Since the integrand is continuous at every point of $(0,1)$, differentiation yields

$$0=2t\widehat{x}(t)+t^2,\qquad\text{i.e.,}\quad\widehat{x}(t)=-t/2.$$

The only values of A and B compatible with this arc are $A=0$, $B=-1/2$. For any other choices of the endpoint conditions, there is no admissible extremal, so the problem can have no solution.

With $A=0$ and $B=-1/2$, however, $\widehat{x}(t)=-t/2$ is the unique global minimizer. To prove this, complete the square to rewrite the integrand as

$$L(t,x,v)=tx^2+t^2x=t(x+t/2)^2-t^3/4.$$

Then for any admissible arc x, one has

$$\begin{aligned}\Lambda[x]-\Lambda[\widehat{x}]&=\int_0^1\left[t\left(x(t)+t/2\right)^2-t^3/4-t\left(\widehat{x}(t)+t/2\right)^2+t^3/4\right]\,dt\\&=\int_0^1 t\left(x(t)+t/2\right)^2\,dt\\&\geq 0,\end{aligned}$$

with equality if and only if $t\left(x(t)+t/2\right)^2=0$ for each t in $[0,1]$. (There can be no exceptions, because the integrand is a continuous function by hypothesis.) This forces $x(t)=-t/2=\widehat{x}(t)$ for each t in $(0,1]$, and continuity at $t=0$ then makes x identical to $\widehat{x}$ on the whole closed interval. Hence $\widehat{x}$ gives the problem's unique global minimum.