# UBC Mathematics 402(201)—Assignment 2
## Due by PDF upload to Canvas no later than 23:59, Friday 23 January 2026

**1.** (a) Find the unique admissible extremal $\widehat{x}$ in the problem

$$\min\left\{\int_1^2 \left[2x^2(t)+t^2\dot{x}^2(t)\right]\,dt \;:\; x(1)=1,\ x(2)=5\right\}.$$

(Hint: The Euler equation has two solutions of the form $t^p$.)

(b) Prove that $\widehat{x}$ is the global solution to this problem.

**2.** Find all extremals of

$$\int \left(t^2+x^2(t)\right)^{1/2}\left(1+\dot{x}^2(t)\right)^{1/2}\,dt.$$

(Hint: Switch to polar coordinates $(r,\theta)$. Then choose wisely between the interpretation where $r=r(\theta)$ and the alternative where $\theta=\theta(r)$. Of course, final answers should be expressed in the same variables as the original problem.)

**3.** Let $V_{II}=\left\{x\in C^1[0,1] \,:\, x(0)=0=x(1)\right\}$. Define $\Lambda\colon V_{II}\to\mathbb{R}$ by

$$\Lambda[x]=-\dot{x}(0)^3+\int_0^1 \dot{x}(t)^2\,dt.$$

(a) Given any $\widehat{x}\in V_{II}$ and $h\in V_{II}$, define $\phi(\lambda)=\Lambda[\widehat{x}+\lambda h]$. Evaluate $\phi'(0)$ and $\phi''(0)$ in terms of $\widehat{x}$ and $h$.

(b) Consider the arc $\widehat{x}=0$. Use the information in (a) to show that for any nonzero $h\in V_{II}$, the function $\phi$ has a strict local minimum at the point $\lambda=0$. (That is, show that $\widehat{x}=0$ is a directional local minimizer for $\Lambda$.)

(c) Construct a sequence of arcs $y_n$ in $V_{II}$ with $\Lambda[y_n]<\Lambda[0]$ for all $n$, even though

$$\max_{t\in[0,1]}|y_n(t)|\to 0 \quad\text{and}\quad \max_{t\in[0,1]}|\dot{y}_n(t)|\to 0 \qquad\text{as}\quad n\to\infty.$$

Hence $\widehat{x}=0$ is a directional local minimizer, but not a "weak local minimizer".
[Clue: Try arranging $\dot{y}_n$ so that the integral is very small, but $\dot{y}_n(0)=1/n$.]

**4.** Consider this extension of the Basic Problem, in which the interval $[a,b]$ is given, but the endpoint values of the competing arcs $x$ are free to vary:

$$\text{minimize } \Lambda[x]=k(x(a))+\ell(x(b))+\int_a^b L(t,x(t),\dot{x}(t))\,dt$$
$$\text{over all}\quad x\in PWS[a,b].$$

(Assume the given functions $k=k(x)$, $\ell=\ell(x)$, and $L=L(t,x,v)$ are all $C^1$.)

Suppose $\widehat{x}\in PWS[a,b]$ gives the minimum in this problem. Prove that $\widehat{x}$ satisfies not only (IEL), but also the endpoint conditions

$$\widehat{L}_v(a^+)=k'(\widehat{x}(a)),\quad \widehat{L}_v(b^-)=-\ell'(\widehat{x}(b)).$$

**5.** Consider the problem of minimizing $\Lambda$ over $PWS[0,1]$ (no endpoint restrictions!), given

$$\Lambda[x] = \int_0^1 \left[\tfrac{1}{2}\dot{x}^2(t) + x(t)\dot{x}(t) + \dot{x}(t) + x(t)\right]\,dt.$$

(a) Find the unique extremal satisfying the natural boundary conditions: call it $z$.

(b) Show that $\inf\left\{\Lambda[x] \,:\, x \in PWS[0,1]\right\} = -\infty$. Deduce that no global minimum exists, so the arc $z$ just found cannot be a global minimizer.

(c) Show that $z$ cannot even be a [weak] local minimizer, by proving that the function of two variables defined as follows does not have a local minimum at the origin: $f(m,b) = \Lambda[x(\cdot;m,b)]$, where $x(t;m,b) = z(t) + mt + b$.