UBC Mathematics 402(201)—Assignment 3
Due by PDF upload to Canvas no later than 23:59, Friday 30 January 2026

1. Find the unique admissible extremal for the following instance of the Basic Problem:

$$\min\left\{\Lambda[x]=\int_0^1 \frac{1+x(t)^2}{\dot{x}(t)^2}\,dt \;:\; x(0)=1, x(1)=\cosh(1)\right\}.$$

2. Consider the cycloid whose parametric equations are

$$t(\theta)=R(\theta-\sin\theta),\;\; x(\theta)=R(1-\cos\theta), \qquad 0\le\theta\le\Theta.$$

Here the radius $R>0$ and the terminal angle $\Theta\in(0,2\pi)$ are fixed in advance, and gravity acts in the direction of the *positive* x-axis.

(a) Show that a point mass falling along this curve (starting from rest, without friction) will complete its journey in time $T=T(R,\Theta)=\Theta\sqrt{R/g}$.

(b) Notice that if the bead's terminal point is (b,B), then the contants R and Θ above must satisfy

$$\frac{1-\cos\Theta}{\Theta-\sin\Theta}=\frac{B}{b}, \qquad R=\frac{b}{\Theta-\sin\Theta}=\frac{B}{1-\cos\Theta}.$$

Prove that if $B>0$ and $b>0$ are given, the first of these equations has at least one solution for Θ in $(0,2\pi)$. (Hint: IVT.)

(c) Prove that if $B>0$ and $b>0$ are given, then the solution Θ described in part (b) is *unique.*

(d) Given $b=3$, $B=1$, calculate R, Θ, and T to five significant figures.

(e) Find a formula for the bead's fall time if the path from $(0,0)$ to (b,B) is a straight line. Evaluate the resulting fall time when $b=3$, $B=1$; by what percentage does it exceed the value for the cycloidal path?

[Hint: Identities like $1-\cos\theta=2\sin^2(\theta/2)=2-2\cos^2(\theta/2)$ may help with some integrals.]

3. The flight path of an aircraft is required between two points separated by a distance D on a level desert. The cost of flying a distance ds at height x is $e^{-x/H}\,ds$, where $H>0$ is a given constant. Formulate the corresponding basic problem in the calculus of variations, and identify the only possible candidate for the minimum-cost trajectory.

4. A taut string on the *3D surface* $y=x^2$ runs from the origin to the point $P(2/3,4/9,10/9)$.

(a) Find the function $z(t)$ that completes the parametric expression $(t,t^2,z(t))$, $0\le t\le\frac{2}{3}$, for the points on the string.

(b) Prove the inequality below, and use it to explain why the arc z found in part (a) gives the unique global minimizer for the variational problem considered there.

$$\forall k>0, v_0, v, \qquad \sqrt{k^2+v^2}\ge\sqrt{k^2+v_0^2}+\frac{v_0}{\sqrt{k^2+v_0^2}}(v-v_0).$$