UBC Mathematics 402(201)—Assignment 4
Due by PDF upload to Canvas no later than 23:59, Friday 06 February 2026

1. Consider the problem

$$\min\left\{\Lambda[x]=\int_0^b \sqrt{x(t)\,(1+\dot{x}^2(t))}\,dt \,:\, x(0)=1,\ x(b)=B\right\}.$$

(a) Find a one-parameter family of extremals x passing through the given initial point.

(b) Consider target points (b,B) where both $b>0$ and $B>0$. Show that the number of extremals ending at (b,B) could be 0, 1, or 2. Carefully describe the first-quadrant regions in the (t,x)-plane where each outcome occurs; include a diagram. (See also part (c).)

(c) Take $b=1$, $B=5/2$. Show that two extremals pass through this point, and evaluate their Λ-values. Sketch both extremals on the diagram started in part (b).

2. Consider the variational integrand

$$L(t,x,v)=(v^2+x^2)(x-t)^2-\frac{4}{3}x^3+2tx^2.$$

(a) Show that both functions $x(t)=t$ and $x(t)=\alpha e^t$ satisfy (DEL), for any constant α.

(b) Find all possible corner points for extremals of L.

(c) Find an extremal $\widehat{x}$ for the following instance of the Basic Problem:

$$\min\left\{\Lambda[x]=\int_0^1 L(t,x(t),\dot{x}(t))\,dt \,:\, x(0)=0,\ x(2)=e\right\}.$$

Does your arc $\widehat{x}$ lie in $PWS[0,2]$? In $C^1[0,2]$? In $C^2[0,2]$? Give reasons for your answers.

3. Suppose (only) that $P\colon [a,b]\to\mathbb{R}$ is continuous. Prove that the following are equivalent (TFAE):

(a) $\displaystyle\int_a^b P(t)\ddot{h}(t)\,dt=0$ for each "variation" $h\in C^2([a,b])$ satisfying

$$\text{both}\quad h(a)=0=h(b)\quad\text{and}\quad \dot{h}(a)=0=\dot{h}(b).$$

(b) $P(t)=mt+c$ for some constants m and c.

Caution: This problem cannot be solved simply by substituting $\widetilde{h}=\dot{h}$ and citing the lemma presented in class. It's true that this substitution produces variations $\widetilde{h}$ that belong to V_{II}. However, there are many elements of V_{II} that cannot be generated in this way. In principle, probing the given function P using only a subset of V_{II} may not reveal the same information that we can get by using the whole space. Thus it is not obvious that the full strength of DuBois-Reymond's original conclusion is available. It will be necessary to adapt the proof of DuBois-Reymond's Lemma to this new situation.

4. Consider the vector space X of all functions $x\colon [a,b] \to \mathbb{R}$ whose *derivative* $\dot{x}$ is piecewise smooth. Thus $\ddot{x}$ is piecewise continuous on $[a,b]$, and it makes sense to talk about the functional $M\colon X \to \mathbb{R}$ defined by

$$M[x] := \int_a^b L(t, x(t), \dot{x}(t), \ddot{x}(t))\, dt, \quad \text{for all } x \in X.$$

Here the integrand $L(t,x,v,w)\colon [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1.

(a) Evaluate the derivative operator $DM[\widehat{x}]$, in terms of $\widehat{x} \in X$.

(b) Suppose that some $\widehat{x} \in X$ gives a minimum for M relative to the subspace of X defined by the four conditions $h(a) = 0 = h(b)$ and $\dot{h}(a) = 0 = \dot{h}(b)$. Find an integro-differential equation satisfied by $\widehat{x}$.

(c) When both L and $\widehat{x}$ are sufficiently smooth, repeated differentiation reduces the equation in (b) to a fourth-order ODE for $\widehat{x}$. Find this ODE.

(d) Among all curves x in X that join $\alpha = (0,0)$ to $\beta = (1,0)$ and satisfy $\dot{x}(0) = 1$ and $\dot{x}(1) = -1$, find the one that minimizes

$$M[x] := \int_0^1 (\ddot{x}(t))^2\, dt.$$

Be sure to prove that your candidate really gives the minimum. (Hint: Use (c) to find a candidate, then proceed directly.)