UBC Mathematics 402(201)—Assignment 5
Due by PDF upload to Canvas no later than 23:59, Friday 20 February 2026

1. Find the form of the vector-valued extremals for these variational integrals:

(a) $\displaystyle\int \left[\dot{x}(t)^2 + \dot{x}(t)\dot{y}(t) + \dot{y}(t)^2\right]\,dt.$

(b) $\displaystyle\int \left[\dot{x}(t)^2 - \dot{y}(t)^2 + 2x(t)y(t) - 2x(t)^2\right]\,dt.$

2. (a) Given smooth (C^1) functions $L(t,x,v)$ and $G(t,x,v)$, consider the functionals

$$\Lambda[x] := \int_a^b L(t,x(t),\dot{x}(t))\,dt, \qquad \Gamma[x] := \int_a^b G(t,x(t),\dot{x}(t))\,dt.$$

Prove: If there is some constant $\lambda \geq 0$ for which $\widehat{x}$ minimizes the functional $\widetilde{\Lambda}[x] := \Lambda[x] + \lambda\Gamma[x]$ over the set $\widetilde{S} = \{x \in PWS[a,b] \,:\, x(a) = A, x(b) = B\}$, then $\widehat{x}$ also minimizes the functional $\Lambda[x]$ over the set

$$S = \{x \in PWS[a,b] \,:\, x(a) = A,\ x(b) = B,\ \Gamma[x] \leq \Gamma[\widehat{x}]\}\,.$$

(b) Use (a) to solve this problem:

$$\min\left\{\int_0^1 \dot{x}(t)^2\,dt \,:\, x(0) = 0,\ x(1) = 0,\ \int_0^1 e^t x(t)\,dt \leq -1/2\right\}.$$

3. Define $V\colon (0,\infty) \to \mathbb{R}$ by

$$V(\alpha) = \min\left\{\int_0^1 tx(t)\,dt \,:\, x(0) = 0,\ x(1) = 0,\ \int_0^1 \dot{x}(t)^2\,dt = \alpha\right\}.$$

Identify the unique global minimizer for each $\alpha > 0$ (justify these properties), and derive a formula for $V(\alpha)$. Use this formula to evaluate $V'(\alpha)$.

4. Use Lagrange Multipliers to find all candidates for minimality in this problem:

$$\min\left\{\int_0^\pi \dot{x}(t)^2\,dt \,:\, x(0) = 0,\ x(\pi) = 0,\ \int_0^\pi x(t)^2\,dt = 1\right\}.$$

(There are infinitely many.)

5. Consider this problem, in which B is given:

$$\min\left\{\Lambda[x] := \int_0^1 -x(t)\sqrt{1-\dot{x}(t)^2}\,dt \,:\, x(0) = 0,\ x(1) = B\right\}.$$

(a) Explain why there can be no solution if $|B| > 1$.

(b) Assuming $0 < B < 1$ and $x(t) > 0$ for all $t \in (0,1)$, describe the unique candidate for minimality as completely as possible.

(c) Using computer assistance as required, find the unique candidates for minimality in the cases where $B = 1/5, 2/5, 3/5, 4/5$. Report any computed approximations to 5 significant digits, and show the graphs of the four candidates on the same set of (t,x)-axes.