Quadratic Lagrangians

Setup. An interval $[a,b]$ is given, with C^1 functions $\gamma, \beta, \alpha \colon [a,b] \to \mathbb{R}$. These help us define the Lagrangian $L \colon [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via

$$L(t,x,v) = \alpha(t)v^2 + 2\beta(t)xv + \gamma(t)x^2.$$

Introduce the integral functional $\Lambda \colon C^1[a,b] \to \mathbb{R}$ by defining

$$\Lambda[x] = \int_a^b L(t, x(t), \dot{x}(t))\, dt.$$

Expansion. Given arcs $x, h \in C^2[a,b]$ such that $h(a) = 0 = h(b)$, we have*

$$\begin{aligned}
\Lambda[x+h] &= \int_a^b \left(\alpha(t)[\dot{x}+\dot{h}]^2\right)\, dt + 2\beta(t)[x+h][\dot{x}+\dot{h}] + \gamma(t)[x+h]^2 \\
&= \int_a^b \begin{pmatrix} \alpha[\dot{x}^2 + \dot{h}^2 + 2\dot{x}\dot{h}] \\ +2\beta[x\dot{x} + h\dot{h} + x\dot{h} + h\dot{x}] \\ +\gamma[x^2 + h^2 + 2xh] \end{pmatrix} dt \\
&= \Lambda[x] + \Lambda[h] + 2\int_a^b \left(\alpha\dot{x}\dot{h} + \beta[x\dot{h} + h\dot{x}] + \gamma x h\right)\, dt \\
&= \Lambda[x] + \Lambda[h] + 2\int_a^b \left([\gamma x + \beta\dot{x}]h + [\beta x + \alpha\dot{x}]\dot{h}\right)\, dt.
\end{aligned}$$

Integration by parts, using $h(a) = 0 = h(b)$, simplifies the second term in the integral above:

$$\int_a^b [\beta x + \alpha\dot{x}]\dot{h}\, dt = [\beta x + \alpha\dot{x}]\dot{h}\Big|_{t=a}^{b} - \int_a^b h\frac{d}{dt}[\beta x + \alpha\dot{x}]\, dt$$

It follows that

$$\Lambda[x+h] = \Lambda[x] + \Lambda[h] + 2\int_a^b h\left([\gamma x + \beta\dot{x}] - \frac{d}{dt}[\beta x + \alpha\dot{x}]\right)\, dt.$$

Define the *Euler Residual associated with arc* x, $R_x \colon [a,b] \to \mathbb{R}$, like this:

$$R_x(t) = 2[\beta(t)\dot{x}(t) + \gamma(t)x(t)] - 2\frac{d}{dt}\left[\alpha(t)\dot{x}(t) + \beta(t)x(t)\right], \qquad t \in [a,b].$$

For any "variation" h in the calculation above, the scalar multiple λh serves just as well. Substituting λh for h, and exploiting the quadratic behaviour of L and Λ, gives this result:

$$\Lambda[x + \lambda h] = \Lambda[x] + \lambda^2\Lambda[h] + \lambda\int_a^b R_x(t)h(t)\, dt, \qquad \forall r \in \mathbb{R}.$$

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* Here we write x instead of $x(t)$, etc., to save space.

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This expression has a lot to offer. Let's use the notation $\langle f\,,\,g\rangle = \int_a^b f(t)g(t)\,dt$ to condense it:

$$\Lambda[x+\lambda h] = \Lambda[x] + \langle R_x\,,\,h\rangle\,\lambda + \Lambda[h]\lambda^2 =: q(\lambda).$$

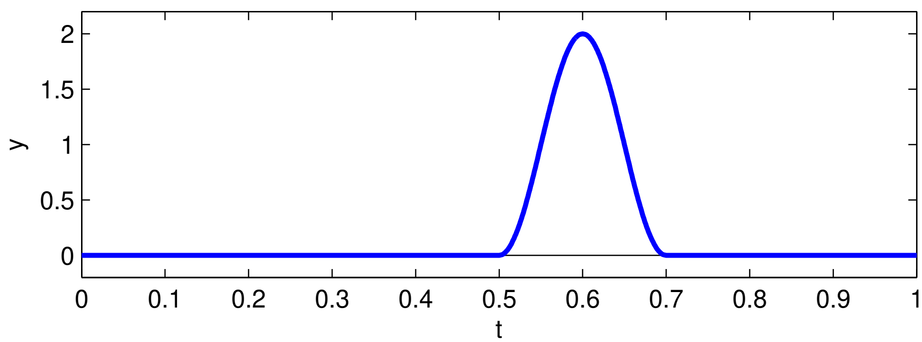
Small Bump Variations. For any nondegenerate subinterval $[a',b']$ of $[a,b]$, consider the arc $h\colon[a,b]\to\mathbb{R}$ defined by

$$h(t) = \begin{cases} 1-\cos\left(2\pi\left[\dfrac{t-a'}{b'-a'}\right]\right), & \text{if } a'\le t\le b',\\ 0, & \text{otherwise.}\end{cases}$$

Notice that $h(t)=0$ for each $t\in[a,a']\cup[b',b]$, so h is a legitimate variation. Moreover, $h\in C^1[a,b]$ because

$$\dot h(t) = \begin{cases} \dfrac{2\pi}{b'-a'}\sin\left(2\pi\left[\dfrac{t-a'}{b'-a'}\right]\right), & \text{if } a'\le t\le b',\\ 0, & \text{otherwise.}\end{cases}$$

(To be sure about $\dot h(a')$ and $\dot h(b')$, notice that both one-sided limit values of $\dot h$ exist and equal 0 for each of these two points; a famous application of the Mean-Value Theorem explains why this implies $\dot h(a')=0=\dot h(b')$.)



Necessary Conditions for Minimality. If the arc x gives the minimum value for Λ among competing functions with the same endpoint values, then for every variation h as above, the quadratic function $q(\lambda)$ must be minimized at $\lambda=0$. Therefore $0=q'(0)=\langle R_x\,,\,h\rangle$. That is,

$$\forall h\ [h(a)=0=h(b)],\qquad \int_a^b R_x(t)h(t)=0.$$

Here the variation h is arbitrary, so the only way this can hold is to have $R_x(t)=0$ for all t. In detail,

$$\frac{d}{dt}\left[\alpha(t)\dot x(t)+\beta(t)x(t)\right] = \beta(t)\dot x(t)+\gamma(t)x(t). \tag{DEL}$$

This is the (differentiated) Euler-Lagrange equation. Note that for any arc x that satisfies (DEL), we have

$$\Lambda[x+h] = \Lambda[x]+\Lambda[h]\,(+0),\qquad \forall h \text{ with } h(a)=0=h(b).$$

Example. When the coefficients γ, β, α are *constant,* (DEL) reduces to

$$\alpha\ddot{x} + \beta\dot{x} = \beta\dot{x} + \gamma x.$$

Suppose $\gamma = 1$ and $\alpha = 1$: then $\ddot{x} - x = 0$ requires $x(t) = c_1 e^t + c_2 e^{-t}$ for some real constants c_1 and c_2. The endpoint conditions $x(a) = A$ and $x(b) = B$ provide a 2×2 system of linear equations that determines the unique compatible values for c_1, c_2. Further, if y is any other arc with $y(a) = A$ and $y(b) = B$, we can define $h = y - x$ and say

$$\Lambda[y] = \Lambda[x + h] = \Lambda[x] + \Lambda[h] \geq \Lambda[x] + 0.$$

(We have $\Lambda[h] \geq 0$ for each and every variation h thanks to the obvious inequality $L(x, v) \geq 0$ for all (x, v).) Conclusion: When $L = v^2 + x^2$, the Basic Problem has a unique global minimum at the arc x identified above.

Remark. In the constant-coefficient case, there is no β in (DEL). Why not? Look at the β-term in the integral we seek to minimize:

$$\int_a^b 2\beta x(t)\dot{x}(t)\,dt = \beta \int_a^b \frac{d}{dt}\left(x(t)^2\right)\,dt = \beta\left[x(b)^2 - x(a)^2\right] = \beta\left[B^2 - A^2\right].$$

So – in the context of the Basic Problem – the term involving β effectively adds a constant to the objective function Λ. This might change the *minimum value*, but it has no effect on the *minimizing input.*

Example. Consider the constant-coefficient case with $\gamma = 1$ and $\alpha = -1$. Then (DEL) says $\ddot{x} + x = 0$, with the general solution

$$x(t) = c_1 \cos(t) + c_2 \sin(t), \qquad c_1, c_2 \in \mathbb{R}.$$

To explore further, assume $a = 0$. Then the endpoint conditions reduce to

$$\begin{aligned} c_1 \qquad\qquad\qquad &= A, \\ c_1\cos(b) + c_2\sin(b) &= B, \end{aligned} \qquad \text{i.e.,} \qquad c_1 = A,\ c_2\sin(b) = B - A\cos(b).$$

In cases where $\sin(b) \neq 0$, a unique solution is available. But if $b = n\pi$ for some integer $n > 0$, we must have $B = A\cos(n\pi) = (-1)^n A$. That is, if $b = n\pi$ then the problem has to be set up just right to have any compatible solution of (DEL) at all, and when it is there is an infinite family of compatible solutions for (DEL),

$$x(t) = A\cos(t) + c_2\sin(t), \qquad c_2 \in \mathbb{R}.$$

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Every different value for $b > 0$ gives a new instance of the Basic Problem, and there appears to be a significant qualitative transition when $b = \pi$. Full understanding will emerge later, but let's fix $b > 0$ and some integer $n \geq 1$, define $\omega_n = n\pi/b$ and let

$h_n(t) = \sin(\omega_n t)$. Note that $h_n(0) = 0$ and $h_n(b) = \sin(n\pi) = 0$. Then calculate

$$\begin{aligned}\Lambda[h_n] &= \int_0^b \left(\dot{h}_n(t)^2 - h_n(t)^2\right)\,dt \\ &= \int_{t=0}^{\pi/\omega_n} \left(\omega_n^2\cos^2(\omega_n t) - \sin^2(\omega_n t)\right)\,dt \\ &= \int_{\theta=0}^{n\pi} \left(\omega_n^2\cos^2\theta - \sin^2\theta\right)\frac{d\theta}{\omega_n} \qquad\qquad (\text{sub } \theta = \omega_n t,\ d\theta = \omega_n\,dt) \\ &= \frac{1}{\omega_n}\left[\omega_n^2\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\right] \\ &= \frac{n\pi}{2\omega_n}\left(\omega_n^2 - 1\right).\end{aligned}$$

Let's focus on the sign of the expression on the right. Two cases arise:

Option 1: If $b < \pi$, then $\omega_n = n\pi/b > n \geq 1$ for each n, so $\Lambda[h_n] > 0$ for each n. This is compatible with the possibility that the unique admissible extremal whose graph connects $(0, A)$ to (b, B) could be a minimizer. (Spoiler: it is!)

Option 2: If $b > \pi$, then $\omega_1 = \pi/b < 1$, so $L[h_1] < 0$. This shows that any admissible solution x of (DEL) is certainly not a minimizer for Λ. (Reason: $\Lambda[x + h_1] = \Lambda[x] + \Lambda[h_1] < \Lambda[x]$.) However, for each integer $n > b/\pi$, we have $\omega_n > 1$ so that $\Lambda[h_n] > 0$. That is, different variations h lead to different behaviours for the quadratic function $\lambda \mapsto \Lambda[x + \lambda h]$: that function will have a maximum value at $\lambda = 0$ when $h = h_1$, and a minimum value at $\lambda = 0$ when $h = h_n$ for any n sufficiently large. Pretty fancy.

Remark. Note that if $m = 2$ and $k = 2$, the integrand just mentioned has

$$L(t, x, v) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}kx^2 = \tfrac{1}{2}mv^2 - V(x), \qquad \text{for } V(x) = \tfrac{1}{2}kx^2.$$

Thus it's compatible with the setup for Hamilton's Principle of Least Action, and our findings when $b > \pi$ suggest that the phrase "least action" is not always appropriate. Well-informed researchers might use the phrase "stationary action" instead, or adopt an interpretation in which the action integral is used only on short-duration segments of the real-world trajectory. ////

Interpretations. Let's return to the Basic Problem (P) for the generic quadratic above, namely, $L(t, x, v) = \alpha(t)v^2 + 2\beta(t)xv + \gamma x^2$. Recall that for any smooth $h = h(t)$ with $h(a) = 0 = h(b)$,

$$\begin{aligned}&\Lambda[x + \lambda h] = \Lambda[x] + \langle R_x\,,\,h\rangle\,\lambda + \Lambda[h]\lambda^2 =: q(\lambda), \text{ where} \\ &R_x(t) = 2[\beta(t)\dot{x}(t) + \gamma(t)x(t)] - 2\frac{d}{dt}\left[\alpha(t)\dot{x}(t) + \beta(t)x(t)\right], \qquad t \in [a, b].\end{aligned}$$

Improvements. When $\Lambda[h] \neq 0$, function q has a unique critical point $\widehat{\lambda}$ defined by

$$0 = q'(\widehat{\lambda}), \quad \text{i.e.,} \quad \widehat{\lambda} = -\frac{\langle R_x\,,\,h\rangle}{2\Lambda[h]},$$

with associated function value

$$q(\widehat{\lambda}) = \Lambda[x] - \frac{\langle R_x\,,\,h\rangle}{4\Lambda[h]}.$$

This general result requires no special hypotheses.

(i) In the case where $\Lambda[h] > 0$, the quadratic $q(\lambda)$ is convex (a.k.a. concave-up), and the critical value above is the lowest result value available on the line in direction h through the point x of function space. This value is achieved by the perturbed arc $x + \widehat{\lambda}h$. If $\widehat{\lambda} \neq 0$, this new arc makes a definite improvement on x.

(ii) In the case where $\Lambda[h] < 0$, the quadratic $q(\lambda)$ is concave (a.k.a. concave-down), and the values of $\Lambda[]$ have no lower bound on the line in direction h through the point x. This implies that the problem of *minimizing* Λ has no solution. The perturbed arc $x + \widehat{\lambda}h$ gives the *maximum* value for Λ on the line in direction h through the point x of function space.

Descent Directions. For variations that are "small" in some suitable sense, the identity above suggests the approximation

$$\Lambda[x+h] - \Lambda[x] \approx \langle R_x\,,\,h\rangle\,.$$

To take a small step from x in a direction that gives a lower Λ-value, any h that makes the right side negative will serve. Every such h is a *descent direction* at x. If R_x is not identically zero, choosing $h \approx -R_x$ will achieve this. The only challenge here is arranging $h(a) = 0 = h(b)$: there is no reason to expect that R_x will have zero endpoint values, so some adaptation may be needed. But, as discussed earlier, if R_x has nonzero value at even one point, there will exist some bump variation h with small support for which $\langle R_x\,,\,h\rangle < 0$.

Stationarity. If the reference arc x is a local minimizer for Λ, there must be no first-order improvements or descent directions. This explains why we must have $\langle R_x\,,\,h\rangle = 0$ for every variation h, and this is the property encapsulated in the Differentiated Euler-Lagrange equation (DEL).

reference

Descent Directions—Example. Consider this problem:

$$\begin{array}{ll} \text{minimize} & \Lambda[x] \stackrel{\text{def}}{=} \displaystyle\int_0^{\pi/2} \left(\dot{x}(t)^2 - x(t)^2\right)\,dt \\ \text{over all} & x \in C^1[0,\pi/2] \\ \text{subject to} & x(0) = 0,\ x(\pi/2) = \pi/2. \end{array}$$

Here $L(t,x,v) = v^2 - x^2$. This fits the pattern above, with $\alpha(t) = 1$, $\beta(t) = 0$, $\gamma(t) = -1$. For an arbitrary admissible arc x, the Euler Residual is

$$R_x(t) = -2x(t) - 2\frac{d}{dt}\left[0 + \dot{x}(t)\right] = -2\ddot{x}(t) - 2x(t).$$

Consider the particular admissible arc $x(t) = t$: this one has

$$R_x(t) = -2t.$$

Since $R_x(t) < 0$ for all $t \in [0, \pi/2]$, we try the small bump variation idea above with subinterval $[a', b'] = [0, \pi/2]$ of full width. That is,

$$h(t) = 1 - \cos(4t), \qquad 0 \le t \le \frac{\pi}{2}.$$

Calculation gives

$$\Lambda[x] = \int_0^{\pi/2} \left(1^2 - t^2\right)\, dt = \frac{\pi}{2}\left[1 - \frac{\pi^2}{12}\right] \approx 0.2789,$$

$$\Lambda[h] = \int_0^{\pi/2} \left(16\sin^2(4t) - (1-\cos(4t))^2\right)\, dt = \frac{13}{4}\pi \approx 10.21$$

$$\langle R_x\,,\, h\rangle = \int_0^{\pi/2} (-2t)\left[1 - \cos(4t)\right]\, dt = -\frac{\pi^2}{4} \approx -2.467.$$

The abstract calculations above show that the greatest reduction in Λ-value by perturbing x by adding some multiple of h comes from choosing the multiple

$$\widehat{\lambda} = -\frac{\langle R_x\,,\, h\rangle}{2\Lambda[h]} = \frac{\pi^2/4}{13\pi/2} = \frac{\pi}{26} \approx 0.1208.$$

The corresponding objective value is

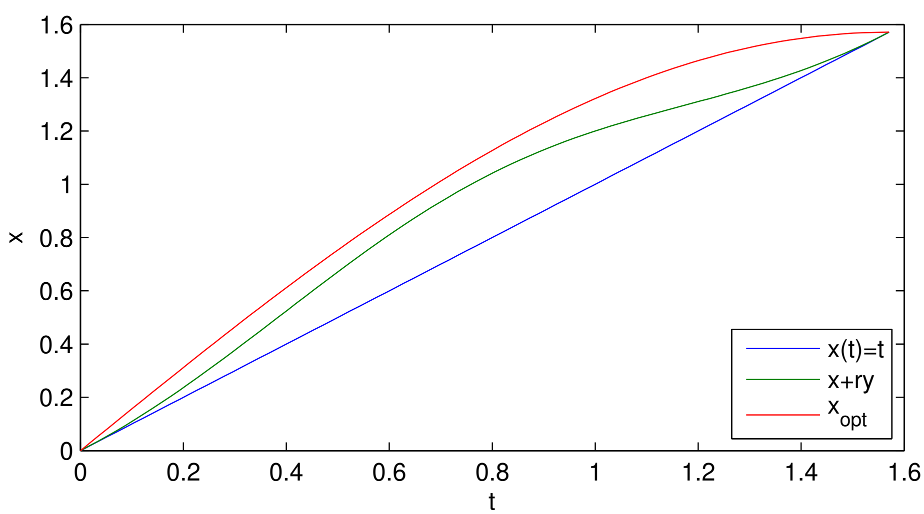
$$\Lambda[x + \widehat{\lambda}h] = q(\widehat{\lambda}) = \frac{\pi}{2} - \frac{29}{624}\pi^3 = 0.1298.$$

This is a rather significant improvement in the objective value, dropping about 53% from the value of $\Lambda[x]$.

The absolute minimizer in this example is known to be

$$\widehat{x}(t) = \frac{\pi}{2}\sin(t), \qquad 0 \le t \le \frac{\pi}{2},$$

with $\Lambda[\widehat{x}] = 0$. Sketches follow (please interpret the caption $x + ry$ as $x + \widehat{\lambda}h$).



Here is another try with the same initial guess. The calculations can all be done by hand, but I used Maple. Take

$$h(t) = t\left(\frac{\pi}{2} - t\right), \qquad 0 \le t \le \frac{\pi}{2}.$$

This has

$$\Lambda[h] = \frac{\pi^3}{24} - \frac{\pi^5}{960} \approx 0.9732,$$
$$\langle R_x\,,\,h\rangle = -\frac{\pi^4}{96}.$$

Consequently the quadratic function $q(\lambda) = \Lambda[x + \lambda h]$ has a global minimum at the point where

$$\widehat{\lambda} = \frac{5\pi}{40 - \pi^2},$$

and the minimum value is

$$\Lambda[x + \widehat{\lambda}h] = q(\widehat{r}) = 0.0144.$$

This is a huge improvement over both the original guess and the previous refinement. Here are some more sketches. (Again, please interpret the caption $x + ry$ as $x + \widehat{\lambda}h$.)