This examination has 5 questions on 2 pages.

The University of British Columbia
Final Examinations—April 2003

Mathematics 403
Stabilization and Optimal Control of Dynamical Systems (*Professor Loewen*)

Open book examination. Time: $2\frac{1}{2}$ hours

Any resources used in class may be used during the examination.
Write your answers in the official examination booklet. Start each solution on a separate page.

[15] **1.** Consider this control system with a scalar input u:

$$(*) \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} u.$$

(i) Find all values of a and b for which system $(*)$ is controllable.

(ii) Assuming $b \neq 0$, find all vectors (v_1, v_2) such that using the feedback law $u = \mathbf{v} \bullet \mathbf{x}$ in $(*)$ produces a system in which every trajectory obeys $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. Give a general solution in terms of a and b, then sketch the appropriate region of the (v_1, v_2)-plane for the special cases $a = 2$, $b = 1$.

[20] **2.** Consider the single-input, control-constrained system

$$\dot{x} = y, \quad \dot{y} = u, \quad |u| \leq 1 \qquad \text{a.e. } t$$

and the target set

$$S = \{(x, y) \,:\, 0 \leq y \leq x + 1,\ x \leq 0\}.$$

For the problem of steering a given initial point in the (x, y)-plane to the target set S, carefully describe the set of initial points for which the minimum-time trajectory ends at a point where $y = x + 1$. Express the corresponding control strategies as functions of position in the (x, y)-plane.

[20] **3.** Find extremal policies for the problem below, where $\delta \in (0, 1)$ and $\xi > 0$ are given:

$$\min \left\{ \int_0^{\pi} e^{-\delta t} \, (u(t) - 1) \, x(t) \, dt \,:\, \dot{x}(t) = u(t) x(t),\ x(0) = \xi,\ u(t) \in [0, 1] \right\}.$$

The cases when δ is sufficiently near 0 and when δ is sufficiently near 1 are qualitatively different. Explain why, and show how to find the δ-value where the change in behaviours takes place. Sketch the extremal state arc x and evaluate $x(\pi)$ in terms of ξ in the special case $\delta = \frac{1}{2}$.

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[25] **4.** Consider this linear-quadratic problem, where x evolves in $\mathbb{R}^n$ and u takes its values in $\mathbb{R}^m$:

$$\min_{u,x} \left\{ \int_0^1 u(t)^T Q u(t)\, dt \;:\; \dot{x} = Ax + Bu,\; x(0) = \xi,\; x(1) = \eta \right\}.$$

There are no control constraints, but it is known that

- the matrix Q is symmetric and positive definite (i.e., $v^T Q v > 0$ whenever $v \neq 0$),
- the dynamical system $\dot{x} = Ax + Bu$ is controllable.

Suppose the function $\widehat{u}$ gives the minimum, with corresponding state $\widehat{x}$.

(i) Show that, in the terminology of the Maximum Principle, $\widehat{u}$ must be a *normal* extremal.

(ii) Show that there must be some constant vector $w \in \mathbb{R}^n$ such that

$$\widehat{x}(t) = e^{At}\xi + \int_0^t e^{A(t-r)} B Q^{-1} B^T e^{-A^T r} w\, dr, \qquad t \in [0,1].$$

(iii) Prove that the vector w described in part (ii) is *unique,* and find a concise formula for calculating it. (The core of this job is proving that a certain symmetric matrix is positive-definite and therefore invertible. Start by showing these properties for Q^{-1}.)

[20] **5.** Consider the following three-dimensional system involving a constant $\varepsilon \geq 0$:

$$\dot{x} = y(1-z), \quad \dot{y} = -x(1-z) - \varepsilon y, \quad \dot{z} = -z(1-z).$$

(i) Prove that the origin is a stable equilibrium point for every $\varepsilon \geq 0$.

(ii) Prove that the origin is asymptotically stable whenever $\varepsilon > 0$.

(iii) Prove that the origin is **not** asymptotically stable when $\varepsilon = 0$. (Clue: Investigate trajectories starting in the (x, y)-plane.)

(iv) Describe the set of initial points $\xi \in \mathbb{R}^3$ for which $\mathbf{x}(t;\xi) \to \mathbf{0}$ as $t \to \infty$.
[Here $\mathbf{x} = (x, y, z)$ and $t \mapsto \mathbf{x}(t;\xi)$ is the system trajectory starting from $\mathbf{x}(0;\xi) = \xi$.]