This examination has 6 questions on 2 pages.

The University of British Columbia
Final Examinations—December 2004
Mathematics 403
Stabilization and Optimal Control of Dynamical Systems (*Professor Loewen*)

Open book examination. Time: $2\frac{1}{2}$ hours

Any resources used in class may be used during the examination.
Write your answers in the official examination booklet. Start each solution on a separate page.

[15] **1.** Consider this linear control system, in which α, β are constant parameters:

$$(1) \qquad \begin{cases} \dot{x}_1 = & & x_2 & + u \\ \dot{x}_2 = & \alpha x_1 & + \beta x_2 & \end{cases}$$

(a) Under what conditions on the constants α, β is system (1) controllable?

(b) Suppose α, β are chosen so that system (1) **is not controllable**. Completely describe the attainable sets $\mathcal{A}(t; \mathbf{0})$ for $t > 0$.

(c) Now suppose α, β are chosen so system (1) **is controllable**. Find constants c and k so that system (1) is equivalent to the following system in canonical form

$$(2) \qquad \begin{cases} \dot{y}_1 & = & & y_2 & \\ \dot{y}_2 & = c & y_1 + k & y_2 & + u. \end{cases}$$

Carefully explain the transformation relating the variables $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

[20] **2.** Consider this nonlinear dynamical system involving a nonzero constant α:

$$\dot{x} = \alpha x^2 - \alpha y, \qquad \dot{y} = 6x - x^2 - xy.$$

(a) Find the unique equilibrium point for this system in the region $x > 0$, $y > 0$. Call it $(\overline{x}, \overline{y})$.

(b) Assess the stability of the equilibrium point $(\overline{x}, \overline{y})$. Express your answer in terms of α.

(c) Let $\alpha = -1$. Find a radius $r > 0$ such that $\mathbb{B}_r(\overline{x}, \overline{y})$ is flow-invariant for the system, and any trajectory starting in $\mathbb{B}_r(\overline{x}, \overline{y})$ converges to $(\overline{x}, \overline{y})$ as $t \to \infty$. Here we have used the notation

$$\mathbb{B}_r(\overline{x}, \overline{y}) = \left\{ (x, y) \,:\, (x - \overline{x})^2 + (y - \overline{y})^2 < r^2 \right\}$$

for the open disk of radius $r > 0$ and centre $(\overline{x}, \overline{y})$.

[20] **3.** Synthesize a feedback control strategy for minimum-time transfer to the origin, given

$$\dot{x}_1 = -3x_1 + 3u, \quad \dot{x}_2 = -x_2 + u, \quad |u| \le 1.$$

That is, use the Maximum Principle to help draw a "map" of the (x_1, x_2)-plane that specifies one u-value for each location. Trajectories generated by the corresponding controls should reach the origin in minimum time.

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[20] **4.** Solve the following problem with $a = 0$, then repeat with $a = 1$:

$$\begin{array}{ll} \text{minimize} & J = \displaystyle\int_0^2 (2x - 3u - \frac{a}{2}u^2)\, dt \\ \text{subject to} & \dot{x} = x + u, \qquad x(0) = 5, \\ & 0 \le u \le 2. \end{array}$$

(*Suggestion*: When $a = 1$, sketch the graph of $w \mapsto H(t, x(t), p(t), w)$ before analyzing extremals.)

[15] **5.** Rocket cars for the model year 2005 have arrived just in time for this test. These frictionless vehicles have twice the thrust of their predecessors:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \qquad |u| \le 2.$$

For a particular starting point $(x_1(0), x_2(0)) = (2, \xi_2)$, and cost criterion

$$J = -x_1(3) + \int_0^3 \left[x_1(t) + \tfrac{1}{2} x_2(t)^2 \right] dt,$$

the system admits a normal extremal control $\widehat{u}$ with the special property that

$$|\widehat{u}(t)| \ne 2 \qquad \forall\, t \in [0, 3].$$

Use this fact to find ξ_2 and the extremal functions $x(\cdot)$, $p(\cdot)$, $\widehat{u}(\cdot)$. (There are no constraints on $x(3)$.)

[15] **6.** Consider the following abstract problem with state x in $\mathbb{R}^n$ and control u in $\mathbb{R}^m$:

$$(*)\quad \left[\begin{array}{ll} \text{minimize} & \ell_0(x(1)) + \displaystyle\int_0^1 L_0(t, x(t), u(t))\, dt \\ \text{subject to} & \ell_j(x(1)) = 0, \quad j = 1, 2, \ldots, M, \\ & \dot{x} = f(t, x, u), \qquad x(0) = \xi, \\ & u \in U. \end{array}\right.$$

Assume the given functions L_0, f, and ℓ_j are continuously differentiable.

(a) What conditions characterize an *abnormal extremal* for this problem?

(b) Prove: if $M = 0$ (i.e., final position constraints are absent), then $(*)$ has no abnormal extremals.

(c) Prove: abnormal extremals cannot exist when $m = n$, $U = \mathbb{R}^n$, $f(t, x, u) = u$, and the set $V = \{\nabla \ell_j(x) \,:\, j = 1, \ldots, M\}$ is linearly independent for each $x \in \mathbb{R}^n$.

(d) Invent a specific instance of $(*)$ for which $n > M$ and there is an abnormal extremal.
(To invent means to demonstrate originality. Don't just copy from your class notes or homework assignments.)

Selected Formulas for Quick Reference

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad\qquad |pq| \le \frac{1}{2}\left(p^2 + q^2\right)$$

$$\dot{x} = Ax + Bu \implies x(t) = e^{A(t-r)}x(r) + \int_r^t e^{A(t-s)} Bu(s)\, ds$$