This examination has 5 questions on 5 pages.

The University of British Columbia
Final Examinations—April 2010

Mathematics 403
Stabilization and Optimal Control of Dynamical Systems (*Professor Loewen*)

Open book examination. Time: $2\frac{1}{2}$ hours

Any resources used in class may be used during the examination.
Write your answers in the official examination booklet. Start each solution on a separate page.

[20] **1.** We explore two feedback strategies, U_1 and U_2, for the frictionless pendulum with forcing:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1) + u.$$

Note that the origin, $(0, 0)$, is an equilibrium point when $u \equiv 0$.

In parts (a)–(b), let $U_1(x_1, x_2) = -\alpha x_2$ for some constant $\alpha > 0$.

(a) Find all equilibrium points for the system when $u = U_1(x_1, x_2)$. Explain why this control cannot drive every trajectory to the origin.

(b) Use a suitable Liapunov function to show that $u = U_1$ makes the origin an asymptotically stable equilibrium point. Estimate the set of initial points $(x_1(0), x_2(0))$ from which this feedback law will drive the trajectory to $(0, 0)$ in the limit as $t \to \infty$.

In parts (c)–(d), let $U_2(x_1, x_2) = \sin(x_1) - \alpha x_1 - \alpha x_2$ for some constant $\alpha > 0$.

(c) Explain why using $u = U_2$ makes the solution from every initial point in the plane converge to $(0, 0)$ as $t \to \infty$.

(d) Find a (quadratic) Liapunov function V that confirms the statement in (c).

Useful Facts

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|pq| \le \frac{1}{2}\left(p^2 + q^2\right), \quad \forall p, q \in \mathbb{R}$$

$$0 \le x - \sin(x) \le \frac{x^3}{6}, \quad \forall x > 0$$

$$\dot{x} = Ax + Bu \implies x(t) = e^{A(t-r)}x(r) + \int_r^t e^{A(t-s)}Bu(s)\,ds$$

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[20] **2.** Consider the following time-varying matrix involving a real constant m:

$$A(t) = \begin{bmatrix} -1 + m\cos^2 t & 1 - m\sin t\cos t \\ -1 - m\sin t\cos t & -1 + m\sin^2 t \end{bmatrix}.$$

(a) Show that the eigenvalues of $A(t)$ are independent of t, and identify the set of m-values for which both eigenvalues have negative real parts.

(b) Show that the time-varying linear system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ has a solution of the form below, where k is some constant defined in terms of m:

$$\mathbf{x}(t) = e^{kt}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}.$$

Identify the set of m-values for which $|\mathbf{x}(t)| \to +\infty$ as $t \to \infty$.

(c) Label each statement below as True or False, and give reasons. Assume all time-varying functions are smooth.

(i) For every constant matrix A in which each eigenvalue has a negative real part, every solution $\mathbf{x}$ of $\dot{\mathbf{x}} = A\mathbf{x}$ obeys $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

(ii) For every matrix-valued function $A = A(t)$ such that each eigenvalue has a negative real part at each t, every solution $\mathbf{x}$ of $\dot{\mathbf{x}} = A(t)\mathbf{x}$ obeys $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

(iii) For every scalar-valued function $a = a(t)$ such that $a(t) < 0$ for each t, every solution x of $\dot{x} = a(t)x$ obeys $x(t) \to 0$ as $t \to \infty$.

[20] **3.** We propose to launch a rocket straight up. Rockets generate thrust by using a chemical reaction to force part of their initial mass downward, so the rocket state has three time-varying components: z, the vertical position; v, the vertical velocity; and m, the mass. Taking the acceleration of gravity to be the constant g, we arrive at the system

$$\frac{dz}{dt} = v, \qquad \frac{dv}{dt} = \frac{u}{m} - g, \qquad \frac{dm}{dt} = -ku. \tag{1}$$

At launch, the rocket has altitude 0 [m], velocity 0 [m/s], and mass $m_0 + m_1$ [kg]:

$$z(0) = 0, \quad v(0) = 0, \quad m(0) = m_0 + m_1. \tag{2}$$

Here m_0 is the mass of the rocket shell and m_1 is the mass of the fuel at launch time. Given some time $T > 0$, we seek the function u that will maximize the altitude $z(T)$ while respecting the total-fuel constraint. This leads to the problem below, in which the final time $T > 0$ is fixed:

Maximize $z(T)$

Subject to dynamics (1) with initial conditions (2),

$$u(t) \in [0,1], \text{ a.e. } t \in [0,T],$$

$$\int_0^T u(t)\,dt = \gamma \stackrel{\text{def}}{=} \frac{m_1}{k}.$$

We assume that $T > \gamma$, so that maximum thrust cannot be sustained for the full interval of interest.

(a) Convert to a minimization problem and write the conditions describing an extremal control strategy, $\widehat{u}$.

(b) Show that the problem has no abnormal extremals.

(c) Find the unique extremal control strategy and the corresponding value of $z(T)$.
[*Hint*: $\widehat{u}(t)$ maximizes over $u \in [0,1]$ some function of the form $s(t)u$. Investigate $\dot{s}(t)$.]

[20] **4.** In this problem $\varepsilon > 0$ is a given constant. Given a starting point (x_0, v_0), the object is to choose $T > 0$ and a function $u\colon [0,T] \to \mathbb{R}$ so as to

$$\begin{aligned} &\text{Minimize} \quad \int_0^T (1 + \varepsilon |u(t)|)\, dt \\ &\text{Subject to } \dot{x}_1 = x_2 \quad \text{a.e.,} \quad x_1(0) = x_0, \quad x_1(T) = 0, \\ &\qquad\qquad\quad \dot{x}_2 = u \quad \text{a.e.,} \quad x_2(0) = v_0, \quad x_2(T) = 0, \\ &\qquad\qquad\quad u(t) \in [-1,1] \quad \text{a.e.} \end{aligned}$$

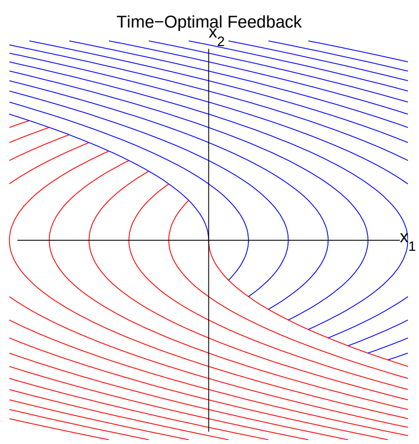
(When $\varepsilon = 0$, the integral simplifies to T, and the problem reduces to the familiar task of steering the rocket car to the origin of phase space in minimum time. But now $\varepsilon > 0$, and its role is to discourage excessive use of control input.)

(a) Derive the following general fact, which will be useful later:

$$\arg \max_{w \in [-1,1]} \{rw - |w|\} = \begin{cases} -1, & \text{if } r < -1, \\ 0, & \text{if } -1 < r < 1, \\ 1, & \text{if } r > 1. \end{cases}$$

(b) Some extremal control functions have the value $+1$ on a nondegenerate final interval of the system path. Use the Pontryagin Maximum Principle to describe the most general form of an extremal control with this property. (When $\varepsilon = 0$, this was an important observation in the rocket car solution: either the control was $+1$ forever, or else the final interval was preceded by exactly one interval where the control was -1.)

(c) Construct an optimal feedback strategy for the portion of phase space in which extremals of the type in (b) provide the evolution. The desired result is a replacement for the upper part of this familiar picture of the phase space for the problem with $\varepsilon = 0$:



Specify the equations for any switching curves you find.

Hint: Remember that the preHamiltonian $\widehat{h}(t) \stackrel{\text{def}}{=} H(t, \widehat{x}(t), p(t), \widehat{u}(t))$ is constant along every extremal, with a value determined by the transversality condition appropriate for problems on variable time intervals.

[20] **5.** Consider the forced harmonic oscillator on the interval $[\tau, 2\pi]$ with a free final state. Given $k_1 \geq 0$ and $k_2 \geq 0$ (not both 0), the problem is

$$\begin{aligned}
\text{minimize} \quad & J[u] \stackrel{\text{def}}{=} \frac{k_1}{2}\left(y(2\pi)\right)^2 + \frac{k_2}{2}\left(\dot{y}(2\pi)\right)^2 + \int_\tau^{2\pi} \frac{1}{2}|u(t)|^2\,dt, \\
\text{over all} \quad & u\colon [\tau, 2\pi] \to \mathbb{R} \text{ piecewise continuous} \\
\text{subject to} \quad & \ddot{y}(t) + y(t) = u(t), \ \text{ a.e. } t \in [\tau, T], \\
& y(\tau) = \xi_1, \ \dot{y}(\tau) = \xi_2.
\end{aligned}$$

(a) Identify extremal strategies for this problem.

(b) Write the true Hamiltonian for this problem.

(c) If $V(\tau, \xi)$ denotes the minimum value in the stated problem [and V happens to be continuously differentiable], then it must satisfy a certain first-order partial differential equation (PDE) and boundary condition (BC). Write these down.

(d) In order for the PDE/BC in part (c) to have a solution of the form

$$V(\tau, \xi) = \tfrac{1}{2}a(\tau)\xi_1^2 + b(\tau)\xi_1\xi_2 + \tfrac{1}{2}c(\tau)\xi_2^2,$$

what conditions must be satisfied by the functions a, b, and c?

(e) The answer in (d) involves a system of ordinary differential equations for the unknown functions a, b, and c. Find as many equilibrium points for this system as you can.