This examination has 5 questions on 5 pages.

The University of British Columbia
Final Examinations—December 2012

Mathematics 403
Stabilization and Optimal Control of Dynamical Systems (*Professor Loewen*)

Time: $2\frac{1}{2}$ hours

Standard UBC Examination Rules apply. In addition, please note the following.

You may use any documents written in your own hand, along with printed copies of any UBC-based resources provided on this year's course web page.

You may not use a telephone, calculator, or computer, or any written resources other than those described above.

Write your answers in the official examination booklet. Start each solution on a separate page.

[20] **1.** Consider the following problem:

$$\begin{aligned}
\text{minimize}\quad & \frac{1}{2}x_1(1)^2 + \int_0^1 \frac{1}{2}\left(u(t)^2 + x_2(t)^2\right)\,dt \\
\text{subject to}\quad & \dot{x}_1(t) = x_2(t) && \text{a.e. } t \in [0,1], \\
& \dot{x}_2(t) = u(t) && \text{a.e. } t \in [0,1], \\
& u(t) \in \mathbb{R} && \text{a.e. } t \in [0,1], \\
& (x_1(0), x_2(0)) = (\xi_1, \xi_2), \\
& x_1(1) \in \mathbb{R},\ x_2(1) = 0.
\end{aligned}$$

(a) Find all abnormal extremals, if any.

(b) Show that there is a unique normal extremal, and determine its functional form. Reduce the problem of determining any constants involved to a system of two linear equations in two variables. It is not necessary to solve this system.

(c) Imagine modifying the given problem by replacing the condition "$u(t) \in \mathbb{R}$" by "$|u(t)| \leq 1$". Find all abnormal extremals in this modified problem. (*Note*: No marks will be given for normal extremals!)

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[20] **2.** Consider the control system

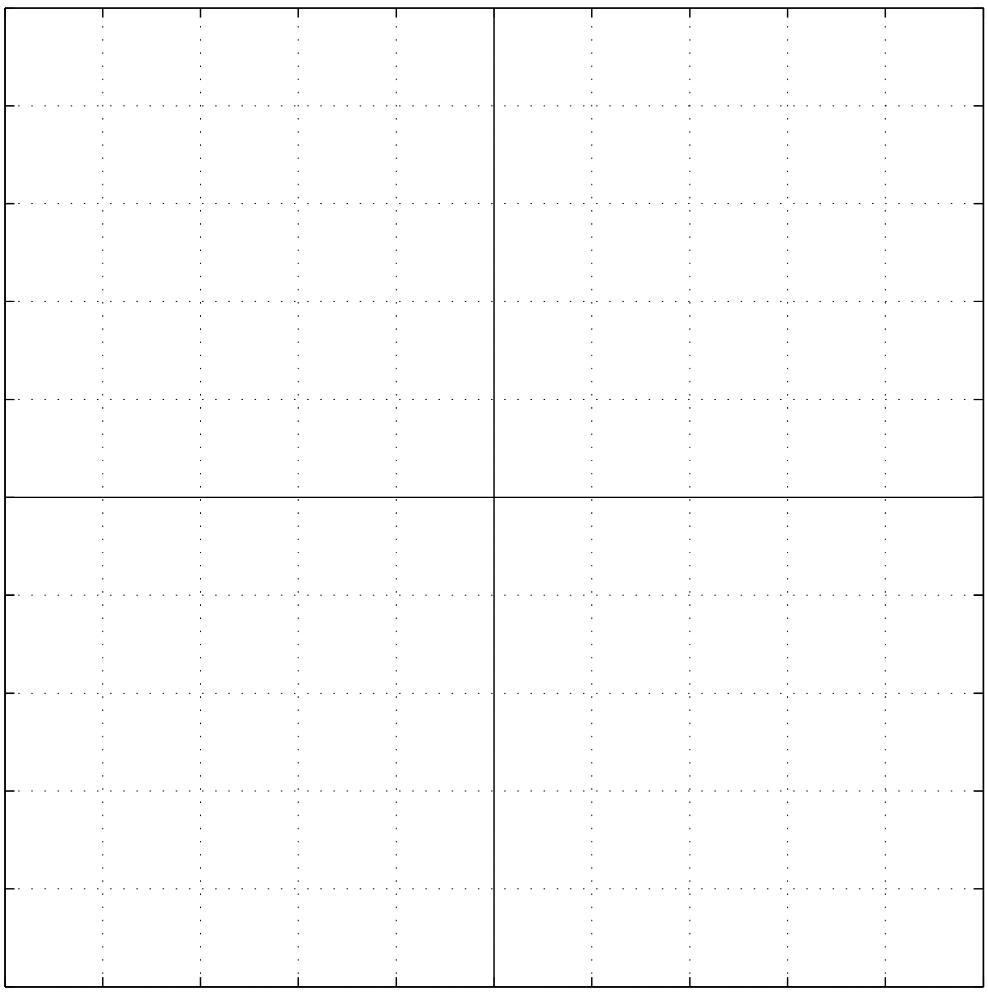
$$\dot{x}_1 = -x_1 + u, \quad \dot{x}_2 = -2x_2 + u, \quad |u| \le 1.$$

Synthesize a feedback control strategy for minimum-time transfer to the target set

$$S = \{(x, 0) \,:\, -1 \le x \le 1\}.$$

That is, use the Maximum Principle to help draw a "map" of the (x_1, x_2)-plane that specifies one u-value for each location. Trajectories generated by the corresponding controls should reach the set S in minimum time.

Write your name on this page and draw your map below. Remember to hand in this entire question package with your exam booklet.



[20] **3.** Consider the controlled nonlinear system

$$(1+x^2)\ddot{x} + bx\dot{x} = u. \tag{$*$}$$

A designer seeks a feedback law for u for which the system state $x(t)$ will follow (or "track") a given input function $\phi = \phi(t)$ satisfying $\ddot{\phi} \equiv 0$, starting from any initial condition. That is, the design objective is to guarantee

$$z(t) \stackrel{\text{def}}{=} x(t) - \phi(t) \to 0 \quad \text{as } t \to \infty. \tag{$\dagger$}$$

(This z is called the "tracking error".)

The designer's idea involves choosing positive constants k and α, and then applying $u = U$, where

$$U(t, x, \dot{x}) = bx\dot{x} - (1+x^2)\left[(k+\alpha)(\dot{x} - \dot{\phi}(t)) + k\alpha(x - \phi(t))\right]. \tag{$**$}$$

(a) Prove that the following function converges to 0 along every trajectory of ($*$)–($**$):

$$V(t, x, \dot{x}) = \left(\dot{x} - \dot{\phi}(t) + \alpha x - \alpha\phi(t)\right)^2.$$

(b) Deduce from (a) that $|z(t)| \le 10^{-3}$ for all t sufficiently large. *Hint*: Conclusion (a) can be expressed as $r(t) \to 0$, where $r(t) = \dot{z}(t) + \alpha z(t)$. So in particular, there must be some T such that $|r(t)| < 10^{-4}\alpha$ whenever $t \ge T$.

(c) [BONUS] Deduce from (a) the desired statement ($\dagger$).

[20] **4.** (Vidale-Wolfe Advertising Model) Our firm supplies a fraction x of industry sales (so $0 \le x \le 1$), from which it collects a gross profit πx. (Here $\pi > 0$ is the price, assumed constant.) Advertising expenditure u affects market share through the equation

$$\dot{x}(t) = au(t)\,[1 - x(t)] - bx(t), \quad x(0) = \xi. \tag{$*$}$$

Here $a > 0$ is a constant indicating the effectiveness of advertising among our non-customers, while $b > 0$ is a constant describiong the rate at which customers are lost to other firms. The present value of our profits over a prescribed planning interval $[0, T]$ consists of gross revenue from sales minus advertising costs:

$$\int_0^T e^{-\delta t}\,[\pi x(t) - u(t)]\; dt.$$

We wish to *maximize* this quantity subject to the given dynamics $(*)$ and the advertising effort constraint $0 \le u \le E$, where $E > 0$ is some constant. (There is no constraint on $x(T)$.)

Assume that a maximizing strategy exists. Using the Maximum Principle, or otherwise, ...

(a) Show that a maximizing strategy requires at most three levels of advertising effort—namely, no effort at all ($u = 0$), maximum effort ($u = E$), and some special intermediate effort $u = u^*$.

(b) Show that if $u(t) = u^*$ on some open interval, then $x(t)$ is constant on this interval. Call the corresponding constant value x^*; give expressions for u^* and x^* in terms of a, b, δ, π.

(c) Assuming $0 < \xi < x^*$, give an informal description of the optimal solution. Full details are not required, but be sure to describe the qualitative behaviour on the final interval correctly.

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[20] **5.** A nonzero constant α is given. A colleague conjectures that for each $\tau < 5$ and $\xi \in \mathbb{R}$, the minimum value in the problem

$$\begin{aligned} &\text{minimize} \quad -x(5) + \int_\tau^5 \frac{u(t)^2}{2}\,dt \\ &\text{subject to } \dot{x}(t) = -\alpha x(t) + u(t) \qquad \text{a.e. } t \in [\tau, 5], \\ &\qquad\qquad\; u(t) \in \mathbb{R} \qquad\qquad\qquad\;\; \text{a.e. } t \in [\tau, 5], \\ &\qquad\qquad\; x(\tau) = \xi, \end{aligned}$$

has the form below, for some constants A, B, C:

$$Ae^{2\alpha t} + Bxe^{\alpha t} + C.$$

Using the Hamilton-Jacobi theory, or otherwise, ...

(a) Prove that the conjecture is correct; identify all compatible choices for A, B, C.

(b) Identify a minimizing control-state pair for the initial point $(\tau, \xi) = (0, 1)$.