The University of British Columbia
Final Examinations—December 2012
Mathematics 403
Stabilization and Optimal Control of Dynamical Systems (*Professor Loewen*)

SOLUTIONS

[20] **1.** Here the endpoint cost is $\ell(x_1, x_2) = \frac{1}{2}x_1^2$ and the preHamiltonian is

$$H(t, x_1, x_2, p_1, p_2, u) = p_1 x_2 + p_2 u - \frac{\lambda_0}{2}\left(u^2 + x_2^2\right).$$

A pair $(\widehat{u}, \mathbf{x})$ is extremal if and only if some $\lambda_0 \in \{0, 1\}$ and arc $\mathbf{p}$, NOT BOTH ZERO, simultaneously satisfy

(TC) $-p_1(1) = \lambda_0 \ell_{x_1}(\mathbf{x}(1)) = \lambda_0 x_1(1)$

(AE) $-\dot{p}_1 = H_{x_1} = 0$, $-\dot{p}_2 = H_{x_2} = p_1 - \lambda_0 x_2$

(HM) $\widehat{u}(t) \in \arg\max_{v\in\mathbb{R}} \left\{ p_2(t)v - \frac{\lambda_0}{2}v^2 \right\}.$

(a) If $\lambda_0 = 0$, then (HM) forces $p_2(t) \equiv 0$ and (TC) forces $p_1(1) = 0$. Using these facts in (AE) makes $p_1(t) \equiv 0$. So the only way that $\lambda_0 = 0$ can happen is if $\mathbf{p} \equiv \mathbf{0}$ also. But the extremality conditions contain a clause that prohibits this. So there are no abnormal extremals.

(b) Assume $\lambda_0 = 1$. We know from (AE) that $p_1(t) \equiv p_1$ is a constant. Also, (HM) implies $\widehat{u}(t) = p_2(t)$, so the given dynamics lead to a chain of first-order differential equations:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \widehat{u} = p_2, \qquad \dot{p}_2 = -p_1 + x_2.$$

These imply that

$$x_1^{(3)} = \ddot{x}_2 = \dot{p}_2 = -p_1 + \dot{x}_1, \quad \text{i.e.,} \quad x_1^{(3)} - \dot{x}_1 = -p_1.$$

The general form of extremals is

$$x_1(t) = A + Be^{-t} + Ce^{t} + p_1 t, \qquad A, B, C \in \mathbb{R},$$

from which we recover

$$x_2(t) = \dot{x}_1 = -Be^{-t} + Ce^{t} + p_1, \qquad p_2(t) = \dot{x}_2 = Be^{-t} + Ce^{t}.$$

Using $x_2(1) = 0$ gives $p_1 = Be^{-1} - Ce^{1}$. Since (TC) says $p_1(1) = -x_1(1)$, we deduce

$$A + 2Be^{-1} + p_1 = 0, \quad \text{so} \quad A = -3Be^{-1} + Ce^{1}.$$

The constants will all be determined once we solve for B and C in the initial conditions:

$$\xi_1 = x_1(0) = A + B + C = (1 - 3e^{-1})B + (1 + e^{1})C,$$

$$\xi_2 = x_2(0) = -B + C + p_1 = (-1 + e^{-1})B + (1 - e^{1})C.$$

The question statement promises full marks for reaching this point. But, just for interest, Maple says

$$B = -\frac{e}{4}\xi_1 - \frac{e}{4}\left(\frac{e+1}{e-1}\right)\xi_2, \qquad C = \frac{1}{4}\xi_1 + \frac{e-3}{4(e-1)}\xi_2.$$

(c) With $\lambda_0 = 0$ in the constrained situation, we still get $p_1(t) \equiv 0$ from (AE) and (TC). Therefore $\dot{p}_2(t) = 0$ and $p_2(t) \equiv k$ for some k. We must have $k \neq 0$ for nontriviality, so p_2 never changes sign. Now the Hamiltonian Maximization condition requires $\widehat{u}(t) = \operatorname{sgn}(p_2)$, so $\widehat{u}$ must be a constant, with value $\widehat{u} = \pm 1$. This choice will satisfy all extremality conditions, provided the corresponding trajectory is feasible. So there are two abnormal scenarios. If $\widehat{u} \equiv \sigma$ with $\sigma^2 = 1$, the dynamics and the endpoint conditions give $x_2(t) = \sigma(t-1)$, $x_1(t) = \frac{1}{2}\sigma(t-1)^2 + C_1$ where $C_1 \in \mathbb{R}$ is arbitrary. The initial conditions require $\xi_1 = \frac{1}{2}\sigma + C_1$, so $C_1 = \xi_1 - \frac{1}{2}\sigma$. But we also need $\xi_2 = x_2(0) = -\sigma$. In short, abnormal extremals occur when the initial point lies on one of the horizontal lines $x_2 = \pm 1$ in the (x_1, x_2)-plane. Intuitively, this is because an initial point on one of these lines leaves no choice of the control that will satisfy the given endpoint conditions: the lone admissible control will automatically be a minimizer for any objective function, not just the one shown here.

[20] **2.** The preHamiltonian for these dynamics is

$$H(x_1,x_2,p_1,p_2,u) = p_1\,(-x_1+u) + p_2\,(-2x_2+u) = -x_1p_1 - 2x_2p_2 + (p_1+p_2)u.$$

Imagine an initial state is given, and the corresponding extremal control-state pair $\widehat{u}(\cdot)$, $\mathbf{x}(\cdot)$ is such that $\mathbf{x}(t)$ first hits the set S at time T. The extremality conditions imply that some costate arc $\mathbf{p}(\cdot)$ obeys

(AE) $-\dot{p}_1(t) = H_{x_1}(\mathbf{x}(t),\mathbf{p}(t),\widehat{u}(t)) = -p_1(t)$, so $p_1(t) = C_1e^{t-T}$ for some $C_1 \in \mathbb{R}$;
$-\dot{p}_2(t) = H_{x_2}(\mathbf{x},\mathbf{p}(t),\widehat{u}(t)) = -2p_2(t)$, so $p_2(t) = C_2e^{-2(t-T)}$ for some $C_2 \in \mathbb{R}$.

(HM) $\widehat{u}(t) \in \arg\max\{[3_1(t)+p_2(t)]v \,:\, -1 \le v \le 1\}$, so $\widehat{u}(t) \in \operatorname{Sgn}(p_1(t)+p_2(t))$ a.e..

(TC) $-(p_1(T),p_2(T)) \in N_S(x_1(T),x_2(T)) \setminus \{(0,0)\}$.

The constants have been chosen so that $(p_1(T),p_2(T)) = (C_1,C_2)$.

To simplify the Hamiltonian Maximization condition (HM), define $\psi(t) = p_1(t)+p_2(t) = C_1e^{t-T} + C_2e^{2(t-T)}$. Then $\widehat{u}(t) \in \operatorname{Sgn}\psi(t)$ in (HM). Notice that

$$\psi(t) = 0 \iff C_1e^{t-T} = -C_2e^{2(t-T)} \iff C_1 = -C_2e^{t-T}.$$

If $C_2 \neq 0$, this has at most one solution for t. Case $C_2 = 0$ is possible, but it forces $C_1 \neq 0$ by nontriviality, and $\psi(t) = C_1e^t$ is then a function with no zeros at all. In any case the control function $\widehat{u}$ must be piecewise constant at level -1 or $+1$, with at most one simple jump discontinuity.

On any open interval (a,b) where $\widehat{u}$ is constant, standard ODE-solving work reveals

$$\frac{d}{dt}[x_1-\widehat{u}] = \dot{x}_1 = -[x_1-\widehat{u}] \qquad \Longrightarrow \quad x_1(t) = \widehat{u} + [x_1(a)-\widehat{u}]e^{-(t-a)},$$

$$\frac{d}{dt}\left[x_2-\tfrac{1}{2}\widehat{u}\right] = \dot{x}_2 = -2[x_2-\tfrac{1}{2}\widehat{u}] \quad \Longrightarrow \quad x_2(t) = \tfrac{1}{2}\widehat{u} + [x_2(a)-\tfrac{1}{2}\widehat{u}]e^{-2(t-a)}.$$

Both x_1 and x_2 are monotonic functions of t, with $(x_1(t),x_2(t)) \to (1,\frac{1}{2})\widehat{u}$ as $t \to \infty$. Special initial conditions allow solutions with $x_1(t) \equiv \widehat{u}$ or $x_2(t) \equiv \frac{1}{2}\widehat{u}$, but when both x_1 and x_2 are nonconstant, eliminating t between the equations above gives

$$\left(\frac{x_1(t)-\widehat{u}}{x_1(a)-\widehat{u}}\right)^2 = \left(\frac{x_2(t)-\frac{1}{2}\widehat{u}}{x_2(a)-\frac{1}{2}\widehat{u}}\right). \qquad (**)$$

Thus the system state follows a parabolic path in the (x_1,x_2)-plane, moving toward the point $(\widehat{u},\frac{1}{2}\widehat{u})$. Some sample trajectories are shown in Figure 1.

Trajectories ending in $(-1,1)\times\{0\}$**.** An extremal state trajectory that hits S from above, at a point in $(-1,1)\times\{0\}$, gives a normal vector to S at the impact point that is a positive multiple of $(0,1)$. Using (TC) gives $p_1(T) = 0$, $p_2(T) = -m$, with $m > 0$, so $\psi(t) = p_2(t) = -me^{t-T}$. Hence $\widehat{u}(t) = -1$ on $[0,T]$ for every such extremal. These trajectories travel to S by moving downward in the upper half-plane along parabolas that all share the vertex $(-1,-\frac{1}{2})$.

An extremal state trajectory that hits S from below, at a point in $(-1,1)\times\{0\}$, gives a normal vector to S at the impact point that is a positive multiple of $(0,-1)$. Using (TC) gives $p_1(T) = 0$, $p_2(T) = m$, $m > 0$, so $\psi(t) = p_2(t) = me^{t-T}$. Hence $\widehat{u}(t) = +1$ on $[0,T]$ for every such extremal. These trajectories travel to S by moving upward in the lower half-plane along parabolas that all share the vertex $(1,\frac{1}{2})$.

Trajectories ending at $(1, 0)$**.** Each outward normal to S at the point $(1, 0)$ has the form (r, s) for some $r \geq 0$ and $s \in \mathbb{R}$. Therefore extremal trajectories that terminate at this point must have costates where $C_1 = p_1(T) \leq 0$. Writing

$$\psi(t) = p_1(t) + p_2(t) = e^{2(t-T)} \left[C_1 e^{-t+T} + C_2 \right]$$

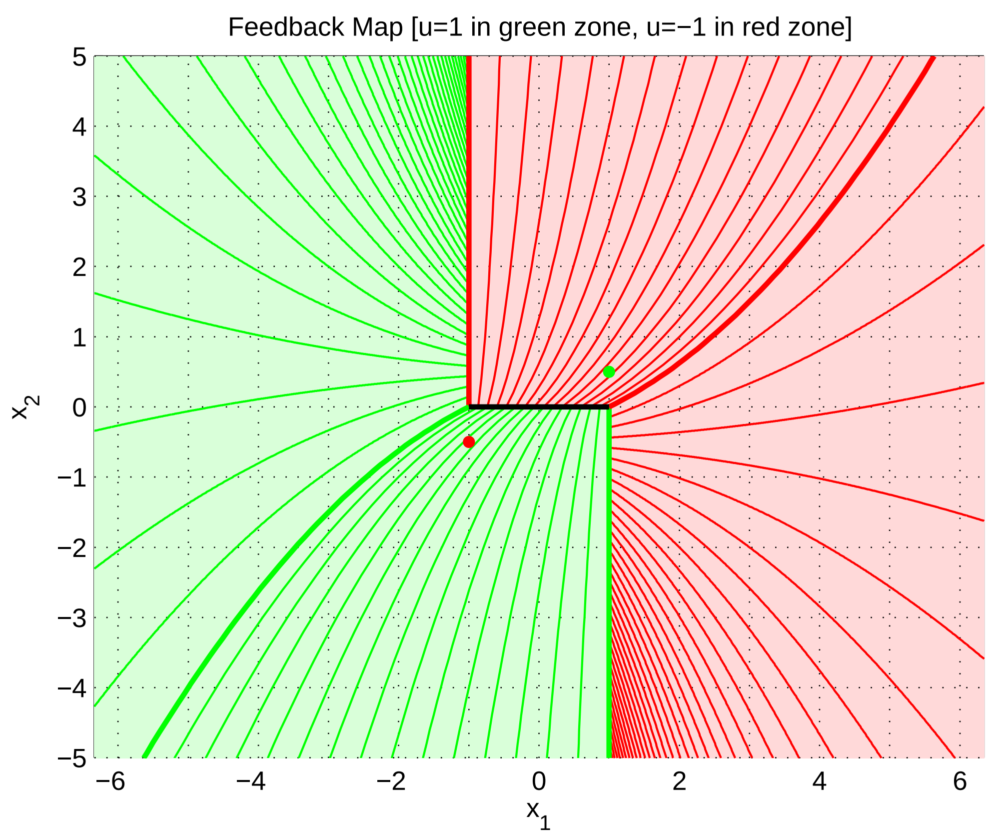
shows a positive term multiplied by a function in brackets that is nondecreasing. So either ψ never changes sign (allowing a constant control of either ± 1) or else ψ changes sign *from negative to positive.* The only state trajectory that hits $(1, 0)$ under the control $\widehat{u} = +1$ satisfies $x_1(t) = 1$ always and $x_2(t) = \frac{1}{2} - \frac{1}{2} e^{-2(t-T)}$. So controls tha reach S at $(1, 0)$ by switching must have a vertical final segment.

Trajectories ending at $(-1, 0)$**.** The situation at the point $(-1, 0)$ in S is the mirror image of the situation at $(1, 0)$. One extremal travels to $(-1, 0)$ by travelling upward along the parabola

$$x_2 = \frac{1}{2} - \frac{1}{8}(x_1 - 1)^2.$$

But for all the extremals that reach this point of S by switching, the switching location has the form $(-1, r)$ for some $r > 0$ and the final segment of the state trajectory is vertical.

Here is the full feedback map. Shading shows the feedback strategy: green where $u = +1$, red where $u = -1$. Some extremal trajectories are also indicated. The target set is drawn in black.



[20] **3.** (a) Plugging $u = U$ into the given system produces cancellation of the term $bx\dot{x}$ and leaves

$$(1+x^2)\ddot{x} = -(1+x^2)\left[(k+\alpha)(\dot{x}-\dot{\phi}(t)) + k\alpha(x-\phi(t))\right].$$

This simplifies to

$$\ddot{x} = -\left[(k+\alpha)(\dot{x}-\dot{\phi}(t)) + k\alpha(x-\phi(t))\right]. \qquad (\ddagger)$$

For any trajectory x of this equation, let

$$v(t) = V(t, x(t), \dot{x}(t)).$$

Then, by the Chain Rule,

$$\dot{v} = 2\left(\dot{x} - \dot{\phi}(t) + \alpha x - \alpha\phi(t)\right)\left[\ddot{x} - \ddot{\phi}(t) + \alpha\dot{x} - \alpha\dot{\phi}(t)\right].$$

Using $\ddot{\phi} \equiv 0$ and the given dynamics for x, we find

$$\begin{aligned}\dot{v} &= 2\left(\dot{x} - \dot{\phi}(t) + \alpha x - \alpha\phi(t)\right)\left[-(k+\alpha)(\dot{x}-\dot{\phi}) - k\alpha(x-\phi) + \alpha\dot{x} - \alpha\dot{\phi}(t)\right] \\ &= 2\left(\dot{x} - \dot{\phi}(t) + \alpha x - \alpha\phi(t)\right)\left[-k(\dot{x}-\dot{\phi}) - k\alpha(x-\phi)\right] \\ &= -2kv(t).\end{aligned}$$

Consequently $v(t) = v(0)e^{-2kt}$ converges to 0 as $t \to \infty$.

(b) With $z(t) = x(t) - \phi(t)$ as defined in (†), introduce $r(t) = \dot{z}(t) + \alpha z(t)$: the result in (a) shows that $r(t) \to 0$ as $t \to \infty$. In particular, there must be some $T > 0$ so large that $|r(t)| \leq 10^{-4}\alpha$ whenever $t \geq T$. Now the identity $\dot{z} = -\alpha z + r(t)$ implies that

$$z(t) = e^{-\alpha t}z(0) + \int_0^t e^{-\alpha(t-s)}r(s)\,ds.$$

Whenever $t \geq T$, this implies

$$\begin{aligned}z(t) &= e^{-\alpha t}z(0) + \int_0^T e^{-\alpha(t-s)}r(s)\,ds + \int_T^t e^{-\alpha(t-s)}r(s)\,ds \\ &= e^{-\alpha t}z(0) + e^{\alpha(T-t)}\int_0^T e^{-\alpha(T-s)}r(s)\,ds + \int_T^t e^{-\alpha(t-s)}r(s)\,ds\end{aligned}$$

On the final interval $[T, t]$ in the third term above, we have $|r(s)| \leq 10^{-4}\alpha$ for each s, so

$$\begin{aligned}\left|\int_T^t e^{-\alpha(t-s)}r(s)\,ds\right| &\leq \int_T^t e^{-\alpha(t-s)}|r(s)|\,ds \\ &\leq 10^{-4}\alpha\int_T^t e^{-\alpha(t-s)}\,ds \\ &= 10^{-4}\alpha\left[\frac{e^{-\alpha(t-s)}}{\alpha}\right]_{s=T}^{T} \\ &= 10^{-4}\left[1 - e^{-\alpha(t-T)}\right] < 10^{-4}.\end{aligned}$$

This inequality holds for every $t > T$, while the first two terms in (‡) clearly have limit 0 as $t \to \infty$. The desired conclusion follows from this.

(c) Given any $\varepsilon > 0$, replace the constant 10^{-4} in the argument of part (b) with $\varepsilon/10$. Identical reasoning will show that for some $T > 0$, one has

$$|z(t)| \leq e^{-\alpha t}|z(0)| + Me^{\alpha(T-t)} + \varepsilon/10$$

for $M = \displaystyle\int_0^T e^{-\alpha(T-s)} r(s)\,ds$. It follows that $|z(t)| < \varepsilon$ for all t sufficiently large; since this holds for arbitrary $\varepsilon > 0$, the desired limiting statement holds.

Quick Alternative: With $z(t) = x(t) - \phi(t)$ as given, and $\ddot{\phi} \equiv 0$, we have $\ddot{z} = \ddot{x}$ so equation (‡) above shows

$$\ddot{z} + (k+\alpha)z + k\alpha z = 0.$$

This is linear with constant coefficients, and it has a solution of the form $z(t) = e^{\lambda t}$ if and only if

$$0 = \lambda^2 + (k+\alpha)\lambda + k\alpha = (\lambda + k)(\lambda + \alpha).$$

Thus the general solution has the form

$$\begin{aligned} z(t) &= C_1 e^{-kt} + C_2 e^{-\alpha t}, && C_1, C_2 \in \mathbb{R}, && \text{if } k \neq \alpha, \\ z(t) &= C_1 e^{-kt} + C_2 t e^{-kt}, && C_1, C_2 \in \mathbb{R}, && \text{if } k = \alpha, \end{aligned}$$

In either case, we have $z(t) \to 0$ as $t \to \infty$. This proves both (b) and (c). Of course, it's easy to compute $\dot{z}(t)$ for both cases above, and deduce that $\dot{z}(t) \to 0$ as $t \to \infty$ as well. Consequently $\dot{z}(t) + \alpha z(t) \to 0$ as $t \to \infty$, and this proves (a).

[20] **4.** Here we are maximizing, so the preHamiltonian is

$$H(t,x,p,u) = pau[1-x] - pbx - e^{-\delta t}[u - \pi x] = (ap[1-x] - e^{-\delta t})u - pbx + e^{-\delta t}\pi x.$$

Extremality for $(\widehat{u}, x)$ requires existence of some costate p satisfying

(TC) $-p(T) = 0$

(HM) For $\psi_0(t) = ap(t)[1-x(t)] - e^{-\delta t}$, $\widehat{u}(t) \in \begin{cases} \{0\}, & \text{if } \psi_0(t) < 0, \\ [0,E], & \text{if } \psi_0(t) = 0, \\ \{E\}, & \text{if } \psi_0(t) > 0. \end{cases}$

(AE) $-\dot{p}(t) = H_x(\cdots) = -(a\widehat{u}(t) + b)p(t) + e^{-\delta t}\pi$

As in class, it is convenient to define $q(t) = e^{\delta t}p(t)$, because then $p(t) = e^{-\delta t}q(t)$, and the adjoint equation becomes

$$-e^{-\delta t}\dot{q}(t) + \delta e^{-\delta t}q(t) = -(a\widehat{u}(t) + b)e^{-\delta t}q(t) + e^{-\delta t}\pi.$$

Rearrangement produces a cleaner equation for q:

$$\dot{q}(t) = (a\widehat{u}(t) + b + \delta)q(t) - \pi.$$

It is convenient also to note that (HM) can also be used with the following function in place of ψ_0:

$$\psi(t) = e^{\delta t}\psi_0(t) = aq(t)[1-x(t)] - 1.$$

(a) For ψ to allow any control other than 0 or E on a nonempty open interval, we must have $\psi(t) = 0$ there, so $aq(t) = 1/[1-x(t)]$. Matching derivatives reveals

$$\begin{aligned} a\dot{q}(t) &= \frac{\dot{x}(t)}{[1-x(t)]^2} \\ (a\widehat{u}(t) + b + \delta)aq(t) - a\pi &= \frac{a\widehat{u}(t)[1-x(t)] - bx(t)}{[1-x(t)]^2} \\ \frac{(a\widehat{u} + b + \delta)}{[1-x(t)]} - a\pi &= \frac{a\widehat{u}(t)[1-x(t)] - bx(t)}{[1-x(t)]^2} \end{aligned}$$

Cancellation eliminates $\widehat{u}$ here, leaving

$$\frac{(b+\delta)}{1-x(t)} - a\pi = -\frac{bx(t)}{[1-x(t)]^2} = -\frac{b(x(t)-1+1)}{[x(t)-1]^2}.$$

This can be rearranged to reveal a quadratic equation with constant coefficients to be satisfied by $x(t)$:

$$\pi a(x-1)^2 + \delta(x-1) - b = 0. \qquad (**)$$

Remembering that meaningful values for x are confined to the interval $[0,1]$, we apply the quadratic formula to deduce that

$$x^* = 1 + \frac{-\delta - \sqrt{\delta^2 + 4\pi ab}}{2\pi a} = 1 - \frac{\delta}{2\pi a}\left(1 + \sqrt{1 + \frac{4\pi ab}{\delta^2}}\right).$$

The constant control required to freeze the state at this level must give $\dot{x}(t) \equiv 0$, so

$$0 = au^*[1 - x^*] - bx^* \iff u^* = \frac{bx^*}{a(1 - x^*)}.$$

(c) Since $\xi < x^*$, we expect to use maximum effort $\widehat{u} \equiv E$ on some initial interval to drive the state up to the singular level. This gives the initial-value problem

$$\dot{x} = aE - (aE + b)x, \qquad x(0) = \xi,$$

which has the solution

$$x(t) = \frac{aE}{aE + b} + \left(\xi - \frac{aE}{aE + b}\right) e^{-(aE+b)t}.$$

The initial interval $[0, \tau]$ ends at the point τ where $x(\tau) = x^*$.

We expect to have $x(t) \equiv x^*$ on some intermediate subinterval of $[0, T]$. Let the intermediate interval be $[\tau, \theta]$.

Notice that $q(T) = e^{\delta T} p(T) = 0$ by (TC), so $\psi(T) = -1$. Since ψ is continuous, we must have $\psi(t) < 0$ on the final subinterval $(\theta, T]$ of the planning period. Therefore we will use $\widehat{u}(t) = 0$ on this interval, and the state will satisfy $x(t) = Re^{-bt}$ for some constant R. Assuming $\tau > \theta$ (strictly), we must have $x(\theta) = x^*$. Hence

$$x(t) = x^* e^{-b(t-\theta)}, \qquad \theta \le t \le T.$$

Meanwhile, we have $\dot{q} = (b + \delta)q - \pi$ on (θ, T), with $q(T) = 0$, so

$$q(t) = \frac{\pi}{b + \delta}\left[1 - e^{(b+\delta)(t-T)}\right], \qquad \theta \le t \le T.$$

We can find θ from the switching condition

$$0 = \psi(\theta) = aq(\theta)[1 - x(\theta)] - 1 = \frac{\pi a}{b + \delta}\left[1 - e^{(b+\delta)(\theta-T)}\right][1 - x^*] - 1.$$

An exact expression for $T - \theta$ is available here, but the question statement suggests contenting ourselves with the qualitative features of the solution identified above.

Alternative: Solving for u in the dynamics gives

$$u = \frac{\dot{x} + bx}{a(1 - x)}.$$

Typically, $0 < x < 1$, so the advertising effort constraint $0 \le u \le E$ can be expressed as

$$0 - bx(t) \le \dot{x}(t) \le aE(1 - x(t)) - bx(t), \qquad \text{a.e. } t \in [0, T].$$

Also, the integral we must *minimize* takes the form

$$\int_0^T e^{-\delta t}\left[\frac{bx(t)}{a(1-x(t))}-\pi x(t)+\frac{\dot{x}(t)}{a(1-x(t))}\right]\,dt.$$

This fits the pattern of the constrained variational problem we studied in class. Here

$$M(x)=\frac{bx}{a(1-x)}-\pi x=\frac{b(x-1)+b}{a(1-x)}-\pi x=\frac{b}{a(1-x)}-\pi x-\frac{b}{a},$$

$$N(x)=\frac{1}{a(1-x)},$$

$$\Delta(x)=M'(x)+\delta N(x)=\frac{b}{a(1-x)^2}-\pi+\frac{\delta}{a(1-x)}=\frac{1}{a(1-x)^2}\left[b+\delta(1-x)-\pi a(1-x)^2\right].$$

We recognize line $(**)$ above as an equivalent form of the condition $\Delta(x)=0$ used to define x^* in our classroom discussion.

[20] **5.** (a) Here the target set S in (t, x)-space is $\{(5, x) : x \in \mathbb{R}\}$, and the preHamiltonian is $H(t, x, p, u) = -\alpha px + pu - \frac{1}{2}u^2$. For given (t, x, p), the preHamiltonian is maximized by the unique element u of $\widehat{U}(t, x, p) = \{p\}$. This gives the true Hamiltonian

$$\mathcal{H}(t, x, p) = -\alpha px + \tfrac{1}{2}p^2.$$

Consider $W(t, x) = Ae^{2\alpha t} + Bxe^{\alpha t} + C$, for which

$$W_t = 2\alpha Ae^{2\alpha t} + \alpha xBe^{\alpha t}, \qquad W_x = Be^{\alpha t}.$$

The function W satisfies the Hamilton-Jacobi Equation $W_t = H(t, x, -W_x)$ if and only if

$$2\alpha Ae^{2\alpha t} + \alpha xBe^{\alpha t} = \alpha xBe^{\alpha t} + \tfrac{1}{2}B^2e^{2\alpha t}.$$

This requires $A = \dfrac{1}{4\alpha}B^2$. Insisting $W(t, x) = \ell(t, x)$ on S amounts to requiring $W(5, x) = -x$ for all x, i.e.,

$$-x = W(5, x) = Ae^{10\alpha} + C + Bxe^{5\alpha}.$$

This entails $C = -Ae^{10\alpha}$ and $B = -e^{-5\alpha}$. We conclude that

$$A = \frac{e^{-10\alpha}}{4\alpha}, \qquad B = -e^{-5\alpha}, \qquad C = -\frac{1}{4\alpha}.$$

As noted in class, the value function is the unique smooth solution of (HJE)/(BC); since W is also a smooth solution, the conjecture must be correct.

(b) Take the feedback law $u = \widehat{U}(t, x, -W_x(t, x)) = e^{\alpha(t-5)}$ and plug into the dynamics:

$$\dot{x} = -\alpha x + e^{\alpha(t-5)}, \qquad x(0) = 1.$$

A particular solution must have the form $x(t) = Ce^{\alpha(t-5)}$ for some constant C. Hence the general solution must have the form $x(t) = Re^{-\alpha t} + Ce^{\alpha(t-5)}$ for some R. Back-substition above reveals the unique trajectory:

$$x(t) = e^{-\alpha t} + \frac{e^{-5\alpha}}{2\alpha}\left[e^{\alpha t} - e^{-\alpha t}\right].$$