Optimal Fishing: Velocity Constraints, Discounting, Singular Arcs

UBC Math 403 Lecture Notes by Philip D. Loewen

We study a Calculus of Variations problem with a scalar-valued state, subject to simple velocity constraints. The time interval $[0,T]$ and the initial state $\xi \in \mathbb{R}$ are fixed, together with a discount rate $\delta \geq 0$ and continuously differentiable functions A, B, ℓ, M, N:

$$\begin{aligned}
\text{minimize} \quad & \ell(x(T)) + \int_0^T e^{-\delta t}\left[M(x(t)) + N(x(t))\dot{x}(t)\right] dt \\
\text{subject to} \quad & A(x(t)) \leq \dot{x}(t) \leq B(x(t)) \quad \text{a.e. } t \in [0,T], \\
& x(0) = \xi.
\end{aligned}$$

We assume that the velocity constraints are compatible with any constant state trajectory, i.e.,

$$\forall x \in \mathbb{R}, \qquad A(x) \leq 0 \leq B(x). \tag{A1}$$

Further, we insist that the velocity available at each state should allow a nontrivial choice:

$$\forall x \in \mathbb{R}, \qquad 0 < D(x) \stackrel{\text{def}}{=} B(x) - A(x). \tag{A2}$$

A special state. Since $N(\cdot)$ is continuous, the function

$$S(z) \stackrel{\text{def}}{=} \int_\xi^z N(r)\,dr, \qquad z \in \mathbb{R}$$

obeys $S'(x) = N(x)$ for all x. So along any arc $x(\cdot)$, we have

$$\frac{d}{dt}S(x(t)) = S'(x(t))\dot{x}(t) = N(x(t))\dot{x}(t).$$

Therefore (integrate by parts)

$$\begin{aligned}
\int_0^T e^{-\delta t}N(x(t))\dot{x}(t)\,dt &= \int_0^T e^{-\delta t}\frac{d}{dt}S(x(t))\,dt \\
&= e^{-\delta t}S(x(t))\Big|_0^T - \int_0^T (-\delta e^{-\delta t})S(x(t))\,dt \\
&= e^{-\delta T}S(x(T)) - S(\xi) + \delta\int_0^T e^{-\delta t}S(x(t))\,dt
\end{aligned}$$

Note that $S(\xi) = 0$, so the function we seek to minimize is identical to

$$\ell(x(T)) + e^{-\delta T}S(x(T)) + \int_0^T e^{-\delta t}\left[M(x(t)) + \delta S(x(t))\right] dt.$$

Here the integral term depends only on the state, not on the velocity. This makes the integral, in isolation, very easy to minimize: just hold the state constant at whatever value minimizes the function $M + \delta S$. Since such a number must be a critical point, we focus on

$$\Delta(x) = M'(x) + \delta S'(x) = M'(x) + \delta N(x),$$

and assume that there is a number x^* such that

$$\begin{cases} \Delta(x) < 0, & \text{for } x < x^*, \\ \Delta(x) = 0, & \text{for } x = x^*, \\ \Delta(x) > 0, & \text{for } x > x^*. \end{cases} \tag{A3}$$

These conditions ensure that x^* provides the unique global minimizer for the function $M + \delta S$.

Economic-Style Interpretation. Assume $\delta > 0$. In the limit as $T \to \infty$, one might describe the integral

$$J[x(\cdot)] = \int_0^\infty e^{-\delta t} \left[M(x(t)) + N(x(t))\dot{x}(t) \right] \, dt$$

as the present value of the long-term cost of an arc $x(\cdot)$. Imagine starting at some state x_0, and choosing to keep the state constant at that level forever. The long-term cost would be

$$J[x_0] = \int_0^\infty e^{-\delta t} \left[M(x_0) + 0 \right] \, dt = \frac{M(x_0)}{\delta}.$$

Alternatively, one could postulate a nearly-instant jump from x_0 to x_1 near time 0, and using the constant state x_1 ever afterward. The cost of this plan would be

$$\int_0^{0^+} e^{-\delta t} M(x(t)) \, dt + \int_0^{0^+} e^{-\delta t} N(x(t)) \frac{dx}{dt} \, dt + \int_{0+}^\infty e^{-\delta t} \left[M(x_1) + 0 \right] \, dt$$
$$= N(x_0)[x_1 - x_0] + J[x_1].$$

The first term is the cost of the initial transition; the second is the long-term cost associated with the revised constant state. On the infinitesimal scale, with $x_1 = x_0 + dx$, the suggested change in strategy augments the cost by

$$dJ = J[x_0 + dx] - J[x_0] + N(x_0)dx = \left[\frac{1}{\delta} M'(x_0) + N(x_0) \right] \, dx = \frac{1}{\delta} \Delta(x_0) \, dx.$$

If $x_0 < x^*$, we have $\Delta(x_0) < 0$ and an incremental change $dx > 0$ decreases the long-term penalty, so it's a good idea. If $x_0 > x^*$, we have $\Delta(x_0) > 0$ so an incremental change $dx > 0$ increases the long-term penalty, so it's a bad idea. First-order analysis identifies x^* as the point where the reduction in long-term cost, namely $-\delta^{-1} M'(x) \, dx$, exactly balances the instantaneous cost required to achieve it, namely $N(x) \, dx$.

Control Reformulation. It's easy to capture the velocity constraints above in the notation of optimal control. The idea is to let the "control" u select the location of $\dot{x}$ in the interval $[A(x), B(x)]$. To capture this with efficient notation, define

$$f(x,u) = A(x) + (B(x) - A(x))\,u = A(x) + D(x)u,$$
$$L(x,u) = M(x) + N(x)f(x,u) = M(x) + N(x)\left[A(x) + D(x)u\right]$$

This leads to the following reformulation of our original problem

$$\begin{aligned} \text{minimize} \quad & \ell(x(T)) + \int_0^T e^{-\delta t} L(x(t), u(t))\,dt \\ \text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)) \quad \text{a.e. } t \in [0,T], \\ & u(t) \in [0,1] \quad \text{a.e. } t \in [0,T], \\ & x(0) = \xi. \end{aligned}$$

The corresponding preHamiltonian is

$$\begin{aligned} H(t,x,p,u) &= pf(x,u) - e^{-\delta t} L(x,u) \\ &= (p - e^{-\delta t}N(x))(A(x) + uD(x)) - e^{-\delta t}M(x). \end{aligned}$$

Suppose $\widehat{u}(\cdot)$ is an extremal control. Then there must be some costate arc p satisfying, among other conditions, the adjoint equation:

$$-\dot{p}(t) = p(t) f_x(x(t), \widehat{u}(t)) - e^{-\delta t} L_x(x(t), \widehat{u}(t)).$$

Here the substitution $q(t) = e^{\delta t} p(t)$ is convenient, because

$$\dot{p}(t) = \frac{d}{dt}\left(e^{-\delta t} q(t)\right) = -\delta e^{-\delta t} q(t) + e^{-\delta t}\dot{q}(t) = e^{-\delta t}\left[\dot{q}(t) - \delta q(t)\right].$$

This puts a common factor of $e^{-\delta t}$ into every term of the adjoint equation, which can then be reduced to an autonomous equation for q:

$$\begin{aligned} \dot{q} &= \delta q - q f_x(x, \widehat{u}) + L_x(x, \widehat{u}) \\ &= \delta q - q f_x(x, \widehat{u}) + M'(x) + N'(x) f(x, \widehat{u}) + N(x) f_x(x, \widehat{u}) \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{AE}$$

(Here each of q, x, $\widehat{u}$ is time-dependent.) The maximum condition says

$$\begin{aligned} \widehat{u}(t) &\in \arg\max_{w \in [0,1]} \left\{ p(t) f(x(t), w) - e^{-\delta t} L(x(t), w) \right\} \qquad \text{a.e. } t \in [0,T] \\ &= \arg\max_{w \in [0,1]} \left\{ e^{-\delta t} q(t) f(x(t), w) - e^{-\delta t} L(x(t), w) \right\} \qquad \text{a.e. } t \in [0,T]. \end{aligned}$$

The function of w we must maximize here is linear, thanks to the special structure of f and L. Indeed, dropping terms independent of w above reveals

$$\widehat{u}(t) \in \arg\max_{w \in [0,1]} \left\{ e^{-\delta t} D(x(t)) \psi(t) w \right\} = \begin{cases} 0, & \text{if } \psi(t) < 0, \\ [0,1], & \text{if } \psi(t) = 0, \\ 1, & \text{if } \psi(t) > 0, \end{cases} \tag{MC}$$
$$\text{where} \quad \psi(t) = q(t) - N(x(t)).$$

(Here it's essential that $e^{-\delta t}D(x(t)) > 0$ for each t, so the choice of maximizer depends entirely on the sign of the factor we have used to define ψ.) The time-dependence of $\widehat{u}$ is completely determined by the evolution of ψ, so we differentiate this along the trajectory of interest:

$$\begin{aligned}\dot{\psi}(t) &= \dot{q}(t) - N'(x(t))\dot{x}(t)\\ &= \Big(\delta q - q f_x(x,\widehat{u}) + M'(x) + N'(x)f(x,\widehat{u}) + N(x)f_x(x,\widehat{u})\Big) - N'(x)f(x,\widehat{u})\\ &= \delta N(x) + \delta(q - N(x)) + M'(x) - (q - N(x))f_x(x,\widehat{u})\\ &= (\delta - f_x(x,\widehat{u}))\psi + \Delta(x).\end{aligned} \tag{AAE}$$

Again, note that each of $x, \widehat{u}, q, \psi$ above is the name of a time-dependent function. Note, too, that the information contained in the state-costate pair $(x(\cdot), q(\cdot))$ is equivalent to the information available in $(x(\cdot), \psi(\cdot))$. Thus line (AAE) can be viewed as an alternative to the original adjoint equation.

Let's write $F(t) = \delta - f_x(x(t), \widehat{u}(t))$ and $d(t) = \Delta(x(t))$ temporarily, to express (AAE) in the simpler form $\dot{\psi} = F(t)\psi + d(t)$. For any constant $\theta \in [0, T]$, we deduce

$$\begin{aligned}\psi(t) &= \psi(\theta)\exp\left(\int_\theta^t F(r)\,dr\right) + \int_\theta^t \exp\left(\int_s^t F(r)\,dr\right) d(s)\,ds\\ &= \psi(\theta)\exp\left(\int_\theta^t (\delta - f_x(x(r),\widehat{u}(r)))\,dr\right)\\ &\qquad + \int_\theta^t \exp\left(\int_s^t (\delta - f_x(x(r),\widehat{u}(r)))\,dr\right)\Delta(x(s))\,ds, \qquad t \in [0, T].\end{aligned}$$

Simple Transitions. The maximum condition determines $\widehat{u}(t)$ at each t when $\psi(t) \neq 0$. At any instant θ where $\psi(\theta) = 0$, we have $\dot{\psi}(\theta) = \Delta(x(\theta))$. To proceed, recall assumption ($A3$). If $\psi(\theta) = 0$ and $x(\theta) > x^*$, then $\dot{\psi}(\theta) = \Delta(x(\theta)) > 0$ implies that ψ has a simple sign change from negative to positive at θ, so the control $\widehat{u}$ must make a simple switch from $\widehat{u}(\theta^-) = 0$ to $\widehat{u}(\theta^+) = 1$. In particular, a transition of the opposite form (from $\widehat{u} = 1$ to $\widehat{u} = 0$) is impossible in the region where $x > x^*$. In the following result, we extend this reasoning from the immediate vicinity of a switching instant to the whole planning interval.

Lemma 1. *Let $(\widehat{u}, x)$ be an extremal pair; consider any $\theta \in (0, T)$ when $x(\theta) < x^*$.*

(i) *If $\psi(\theta) > 0$, then $\psi(t) > 0$ for each $t \in [0, \theta)$, so $\dot{x} = B(x)$ on $[0, \theta)$*

(ii) *If $\psi(\theta) < 0$, then $\psi(t) < 0$ for each $t \in (\theta, T]$, so $\dot{x} = A(x)$ on $(\theta, T]$.*

(iii) *If $\psi(\theta) = 0$, then the extremal state trajectory must satisfy*

$$\dot{x}(t) = \begin{cases} B(x(t)), & t \in [0, \theta),\\ A(x(t)), & t \in (\theta, T].\end{cases}$$

Proof. Since both $x(\cdot)$ and $\psi(\cdot)$ are continuous functions, the set of t in $(0,T)$ where $x(t) < x^*$ and $\psi(t) \neq 0$ must be open. In situations (i) and (ii), this open set contains θ, and we can let (a,b) denote the largest open subinterval of $(0,T)$ containing θ on which both properties hold. Maximality has consequences at both ends of the interval:

- at least one of $a = 0$ or $x(a) = x^*$ or $\psi(a) = 0$ must hold, and
- at least one of $b = T$ or $x(b) = x^*$ or $\psi(b) = 0$ must hold.

(i) If $\psi(\theta) > 0$, then $\psi(t) > 0$ throughout (a,b), and this forces $\widehat{u}(t) = +1$ in (a,b). Consequently $\dot{x} = B(x) \geq 0$ on (a,b), which implies $x(a) \leq x(\theta) < x^*$. Now, seeking a contradiction, suppose $a > 0$ and $\psi(a) = 0$. Then our calculations above show that $\dot{\psi}(a) = \Delta(x(a)) < 0$, and this implies that $\psi(t) < \psi(a) = 0$ for all $t > a$ near a. This contradicts the first line above, so it can't happen. The only viable choice in the first bulleted line above is $a = 0$, and this completes the proof.

(ii) Similarly, if $\psi(\theta) < 0$, then $\psi(t) < 0$ throughout (a,b), and this forces $\widehat{u}(t) = 0$ in (a,b). Consequently $\dot{x} = A(x) \leq 0$ on (a,b), which implies $x(b) \leq x(\theta) < x^*$. Now, seeking a contradiction, suppose $b < T$ and $\psi(b) = 0$. Then our calculations above show that $\dot{\psi}(b) = \Delta(x(b)) < 0$, and this implies that $\psi(t) > \psi(b) = 0$ for all $t < b$ near b. This contradicts the first line above, so it can't happen. The only viable choice in the second bulleted line above is $b = T$, and this completes the proof.

(iii) If $\psi(\theta) = 0$ then $\dot{\psi}(\theta) = \Delta(x(\theta)) < 0$, so for any $h > 0$, no matter how small, there exists $h' \in (0,h)$ for which $\psi(\theta - h') > \psi(\theta) > \psi(\theta + h')$. Apply part (i) to the instant $\theta - h'$ and part (ii) to the instant $\theta + h'$ to deduce that

$$\psi(t) > 0 \ \forall t \in [0, \theta - h], \qquad \text{and} \qquad \psi(t) < 0 \ \forall t \in [\theta + h, T].$$

Since this holds for arbitrary $h > 0$, we must have $\psi(t) > 0$ at each point of $[0,\theta)$ and $\psi(t) < 0$ for each t in $(\theta, T]$. The stated conclusion follows. ////

Of course there is a symmetric result for switches that occur when the state is above the critical value. Here it is.

Lemma 2. *Let $(\widehat{u}, x)$ be an extremal pair; consider any $\theta \in (0,T)$ when $x(\theta) > x^*$.*

(i) *If $\psi(\theta) > 0$, then $\psi(t) > 0$ for every $t \in (\theta, T]$, so $\dot{x} = B(x)$ on $(\theta, T]$.*

(ii) *If $\psi(\theta) < 0$, then $\psi(t) < 0$ for every $t \in [0, \theta)$, so $\dot{x} = A(x)$ on $[0, \theta)$.*

(iii) *If $\psi(\theta) = 0$, then the extremal state trajectory must satisfy*

$$\dot{x}(t) = \begin{cases} A(x(t)), & t \in [0, \theta), \\ B(x(t)), & t \in (\theta, T]. \end{cases}$$

Extremal scenarios compatible with our analysis so far allow for either a constant control on the entire planning interval, or else a piecewise constant control with at most one switch. And for a switch to occur, the relationship of the state at the switching instant to the critical level x^* determines the type of transition allowed.

Singular Arcs. The two lemmas above provide a complete description of extremal arcs that have a switch when the state is not at the critical level x^*. Things get interesting when there is some instant $\theta \in (0,T)$ where $\psi(\theta)=0$ and $x(\theta)=x^*$. In this case, $\dot\psi(\theta)=0$ also, and this allows for the possibility that there is a nondegenerate open interval (a,b) on which both $\psi(t)=0$ and $x(t)=x^*$ hold for each t. Such an interval is called *singular* because the maximum condition (MC) fails to select the optimal control there. Instead, the control is a constant, chosen to maintain a constant state:

$$0=\frac{d}{dt}x^*=A(x^*)+D(x^*)\widehat{u}(t) \iff \widehat{u}(t)=u^*\stackrel{\text{def}}{=}-\frac{A(x^*)}{D(x^*)}.$$

Partly Singular State Trajectories. The lemmas above limit the behaviour of the state before it joins the singular trajectory, and after it leaves. The only possible scenarios are ones where there exist intermediate times $0\le a\le b\le T$ such that

$$\begin{cases} \text{for a.e. } t\in(0,a), & \widehat{u}(t) \text{ is constant with value 0 or 1;}\\ \text{for a.e. } t\in(a,b), & \widehat{u}(t) \text{ is constant with value } u^*=-\dfrac{A(x^*)}{D(x^*)};\\ \text{for a.e. } t\in(b,T), & \widehat{u}(t) \text{ is constant with value 0 or 1.}\end{cases} \tag{8}$$

By allowing the degenerate cases $a=0$, $b=a$, and/or $b=T$, line (8) captures a comparatively manageable set of possibilities.

Transversality. Since $q(t)=e^{\delta t}p(t)$, the transversality condition says

$$-q(T)=-p(T)=e^{\delta T}\ell'(x(T)).$$

Therefore $\psi(T)=-e^{\delta T}\ell'(x(T))-N(x(T))$. Exploring the function $x\mapsto -e^{\delta T}\ell'(x)-N(x)$, especially with regard to its value at x^*, may provide useful information about the sign of $\psi(T)$. This, in turn, will determine the value of $\widehat{u}$ on the final interval (b,T). To illustrate, suppose $\ell(x)=kx$ for some constant k, and $\delta T\gg 1$. Then the dominant contribution to $\psi(T)$ will come from the term $-e^{\delta T}\ell'(x(T))=-ke^{\delta T}$. If $k>0$, the control $\widehat{u}$ will be 0 on the final interval. This makes sense: a final cost of $kx(T)$ with $k>0$ makes it desirable to reduce the final value of the state.

Calculation Tips. Given a concrete problem, we start by evaluating the critical state x^* and studying its relationship with the initial state ξ and the function $x\mapsto -e^{\delta T}\ell'(x)-N(x)$ for clues regarding the behaviour of $\widehat{u}$ in the latter part of $[0,T]$. In typical problems, we have $\delta T\gg 1$ and $\xi\ne x^*$, and all three segments of the scenario in (8) are nontrivial, i.e., $0<a<b<T$. One identifies the transition times a and b as follows.

1. If $\xi>x^*$, use $\widehat{u}=0$ so $\dot x=A(x)\le 0$ on $(0,a)$. The known initial value $x(0)=\xi$ provides a unique solution, which allows one to find the smallest $a>0$ such that $x(a)=x^*$. To allow the state to switch onto the singular

trajectory at time a, we must have $\psi(a) = 0$. Note that $\Delta(x(t)) > 0$ for $t \in [0, a)$, so the identity

$$\dot{\psi}(t) = F(t)\psi(t) + \Delta(x(t)),\ t \in [0, a]; \qquad \psi(a) = 0$$

forces $\psi(t) < 0$ for all $t \in (0, a)$. Thus (MC) holds, and this choice is compatible with the definition of extremality.
[If $\xi < x^*$, do something similar, but start with $\widehat{u} = 1$.]

2. Using some mixture of transversality, common sense, or trial-and-error, we can predict the constant value of $\widehat{u}$ for the final interval (b, T). If, for example, we suspect $\widehat{u} = 1$ there, then we must have

$$\dot{x} = B(x) \text{ on } (b, T), \qquad x(b) = x^*,$$

while

$$\dot{\psi}(t) = (\delta - B'(x(t)))\psi + \Delta(x(t)), \qquad \psi(b) = 0.$$

We will have $x(t) > x^*$ on $(b, T]$, so $\Delta(x(t)) > 0$ there and this will ensure that $\psi(t) > 0$ there as well. So (MC) is certain to hold. The only condition left is transversality. In practical problems, one would write the general solutions for the two differential equations above. Each contains an arbitrary constant. Then one would solve for those two constants and the unknown switching time b by applying the algebraic equations

$$x(b) = x^*, \qquad \psi(b) = 0, \qquad \psi(T) = -e^{\delta T}\ell'(x(T)) - N(x(T)).$$

In problems where δT is not large enough, the extremal evolution may have an internal switch ($0 < a = b < T$) or simply a constant control ($0 = a = b < T$ or $0 < a = b = T$). Suitable special cases of the strategy above can be used.

Fishing Application. Let x denote the biomass of a certain fish stock (in kg, say), and suppose its natural dynamics are governed by some function g so that $\dot{x} = g(x)$. (Typically $g(0) = 0$, $g'(0) > 0$, and $g''(x) \le 0$ for all $x \ge 0$.) Now we propose a fish-harvesting scheme over a fixed time interval $[0, T]$. We will choose the time-varying fishing effort u from some known interval $[0, E]$, where E is our maximum fishing effort, and assume that the rate of pulling fish from the sea is jointly proportional to the fish density and the fishing effort. The revised population dynamics are

$$\dot{x}(t) = g(x(t)) - u(t)x(t), \qquad 0 \le u(t) \le E \qquad \text{a.e. } t \in [0, T].$$

(To make this identity between rates consistent, fishing effort u must be measured in normalized units of $1/s$.) If the selling price for fish is π ($/kg) and the cost per unit effort is c ($), the present value of a given fishing strategy is

$$\int_0^T e^{-\delta t}\,(\pi x(t) - c)\,u(t)\,dt.$$

We seek to maximize this. To fit the theory for minimization problems, we recast this as

$$\begin{aligned}
&\text{minimize} \quad \ell(x(T)) + \int_0^T e^{-\delta t}\,(c - \pi x(t))\,u(t)\,dt \\
&\text{subject to } \dot{x}(t) = g(x(t)) - u(t)x(t), \qquad \text{a.e. } t \in [0,T], \\
&\qquad\qquad 0 \le u(t) \le E, \qquad \text{a.e. } t \in [0,T], \\
&\qquad\qquad x(0) = \xi.
\end{aligned}$$

The new ingredients here are the endpoint cost ℓ, which reflects the value of leaving some fish in the sea at the final time T, and the initial stock ξ, which is supposed to be known.

To recognize this problem as an instance of the one treated above, rearrange the dynamic equation as $u = (g(x) - \dot{x})/x$. Then the input restrictions $0 \le u \le E$ capture the simple velocity constraints of the model above:

$$0 \le \frac{g(x) - \dot{x}}{x} \le E \iff g(x) - Ex \le \dot{x} \le g(x).$$

The same substitution transforms the integrand above as follows:

$$(c - \pi x)\,u = (c - \pi x)\left(\frac{g(x) - \dot{x}}{x}\right) = \left(\frac{c}{x} - \pi\right) g(x) + \left(\pi - \frac{c}{x}\right)\dot{x}.$$

Thus we have all the ingredients above, with

$$\begin{aligned}
A(x) &= g(x) - Ex, \\
B(x) &= g(x), \\
M(x) &= \left(\frac{c}{x} - \pi\right) g(x) \\
N(x) &= \pi - \frac{c}{x}.
\end{aligned}$$

The distinguished level of fish biomass is the unique zero x^* of

$$\begin{aligned}
\Delta(x) &= M'(x) + \delta N(x) \\
&= \left(-\frac{c}{x^2} g(x) + \left(\frac{c}{x} - \pi\right) g'(x)\right) + \delta\left(\pi - \frac{c}{x}\right) \\
&= \frac{1}{x}\left[(\pi x - c)(\delta - g'(x)) - c\left(\frac{g(x)}{x}\right)\right].
\end{aligned}$$

(Unfortunately, the letter u used to denote fishing effort in setting up this model here is re-used *with a different interpretation* in the general discussion above.)

A Specific Case. Logistic population dynamics with natural equilibrium at level $R > 0$ would involve some constant $\gamma > 0$ as follows:

$$g(x) = \gamma x(R - x).$$

The region of interest is the set where $0 < x < R$, i.e., the fish stock is somewhere between extinction ($x = 0$) and the environmental carrying capacity or natural stable equilibrium level ($x = R$). In this case,

$$\begin{aligned} x\Delta(x) &= (\pi x - c)(\delta - \gamma R + 2\gamma x) - c\gamma(R - x) \\ &= 2\pi\gamma x^2 - [\gamma(\pi + cR) - \delta\pi]\, x - c\delta \end{aligned}$$

In the region where $x > 0$, this has a simple sign change from negative to positive at the critical level

$$x^* = \frac{\gamma(\pi + cR) - \pi\delta + \sqrt{[\gamma(\pi + cR) - \delta\pi]^2 + 8\pi\gamma c\delta}}{4\pi\gamma}.$$

This is compatible with assumption $(A3)$ above; here $D(x) = Ex > 0$ for each $x > 0$, so $(A2)$ also holds. Assumption $(A1)$, while realistic, is linked to the environmental issues that trouble the modern world: we have $B(x) = g(x) > 0$ in the region of interest, so stock tends to grow when left alone, but our available fleet is so large that

$$E \geq \gamma, \qquad \text{so} \qquad A(x) = g(x) - Ex = -(E - \gamma)x - \gamma R x^2 < 0 \quad \forall x > 0.$$

That is, we can pull fish from the sea fast enough to drive the stock to extinction.

[*Private notes have further details on a specific numerical example.*]