Stability of Equilibria in Dynamical Systems

Lecture notes for UBC MATH 403

Stability is a fundamental consideration in all kinds of applications. In this section, we study the stability of equilibria in autonomous initial-value problems of the following form:

$$\dot{x} = F(x), \quad x(0) = \xi. \tag{S.1}$$

Later, we will apply the results to controlled systems $\dot{x} = f(x, u)$. We assume that the mapping F is defined and continuously differentiable on some open subset $G \subseteq \mathbb{R}^n$. It then follows from Theorem A.1 that for any initial point $\xi \in G$, the initial-value problem $(S.1)$ has a solution $x(\cdot;\xi)$ valid on some maximal open interval $I(\xi) = (a(\xi), b(\xi))$ containing 0. Here maximality allows either $a(\xi) = -\infty$ or $b(\xi) = +\infty$ or both. When G is a bounded set and F is bounded on G, every initial point ξ for which $b(\xi) < +\infty$ produces a trajectory for which $x(t;\xi)$ converges as $t \to b(\xi)^-$, and the limit point satisfies

$$\lim_{t \to b(\xi)^-} x(t;\xi) \in \operatorname{bdy} G.$$

The standard terminology for equilibrium points and stability includes the following elements.

S.1. Definition. An open set $G \subseteq \mathbb{R}^n$ and a continuously differentiable function $F\colon G \to \mathbb{R}^n$ are given.

(a) A point $z \in G$ is an *equilibrium point (for F)* if $x(t;z) = z$ is a trajectory for F, i.e., if $F(z) = 0$.

(b) A subset $M \subseteq G$ is *flow-invariant (for F)* if, for every $\xi \in M$, both $b(\xi) = +\infty$, and $x(t;\xi) \in M$ for all $t \geq 0$.

(c) An equilibrium point z for F is called *stable* if for every choice of $\varepsilon > 0$, no matter how small, there exists a flow-invariant set M satisfying the inclusions

$$z \in \operatorname{int} M, \ \ M \subseteq \mathbb{B}(z;\varepsilon).$$

(If an equilibrium point z is not stable, we call it *unstable*.)

(d) Let z be an equilibrium point for F. Then a subset $E \subseteq \mathbb{R}^n$ is called a *domain of attraction for z (relative to F)* if E is flow-invariant and has the additional property that for every $\xi \in E$, the solution $x(\cdot;\xi)$ satisfies $x(t;\xi) \to z$ as $t \to \infty$.

(e) An equilibrium point z is called *asymptotically stable* if for every choice of $\varepsilon > 0$, no matter how small, there exists a domain of attraction E for z satisfying both $z \in \operatorname{int} E$ and $E \subseteq \mathbb{B}(z;\varepsilon)$.

Notice that a domain of attraction is a flow-invariant set with additional properties, so every equilibrium point that is asymptotically stable is automatically stable in the ordinary sense. The converse is false, of course, as our first example shows.

S.2. Examples.

(i) $F(x) := 0$, $G = \mathbb{R}^n$. Every point $z \in \mathbb{R}^n$ is an equilibrium point. All are stable, since every subset of $\mathbb{R}^n$ is flow-invariant. However, none are asymptotically stable because the largest domain of attraction of any point z is simply $E = \{z\}$.

(ii) $F(x) := x(1-x)$, $G = \mathbb{R}$. The equilibrium points are $z_0 = 0$, $z_1 = 1$. The point z_1 is asymptotically stable; every open interval of the form $(1 - 1/n, 1 + 1/n)$, $n \in \mathbb{N}$, is a domain of attraction for z_1. However, the point z_0 is unstable: to see how definition S.1(c) fails, imagine any set M with $z_0 = 0$ as an interior point. Then M must contain some negative real number ξ (perhaps small), and the trajectory starting from that ξ will move away from z_0, not toward it.

Note that in this example, the set $(-\infty, 1)$ does not quite qualify for the label "flow-invariant," because $b(\xi)$ is finite for each $\xi < 0$.

(iii) $F(x) = Ax$ for $A \in \mathbb{R}^{n\times n}$, $G = \mathbb{R}^n$. The set $M = \mathbb{R}^n$ is flow-invariant by Thm. A.1. The origin ($z = 0$) is an equilibrium point for F. Its stability depends on the eigenstructure of A.

- If $\Re e(\lambda) > 0$ for some eigenvalue λ of A, then Theorem B.4(b) implies that any set M satisfying $0 \in \operatorname{int} M$ must contain an initial point ξ for which $|x(t;\xi)| \to \infty$ as $t \to \infty$. In particular, any flow-invariant set M satisfying $0 \in \operatorname{int} M$ cannot be bounded, so the origin is an unstable equilibrium point.
- If $\Re e(\lambda) \le 0$ for all eigenvalues λ of A, but some eigenvalue of A has zero real part, then the origin may or may not be stable. If every eigenvalue whose real part vanishes is simple, then the origin is stable; if some such eigenvalue has algebraic multiplicity 2 or bigger, the origin is unstable. In either case, however, the origin cannot be asymptotically stable. To see why, note that Thm. B.4(c) asserts that there are initial points ξ arbitrarily near to 0 for which the solutions $x(\cdot;\xi)$ fail to converge to 0. Any domain of attraction for 0 must fail to contain such points, so it must fail to contain 0 in its interior.
- If $\Re e(\lambda) < 0$ for all eigenvalues λ of A, then Thm. B.4 asserts that $M = \mathbb{R}^n$ is a domain of attraction for $z = 0$. Later in this section we will show that the origin lies at the centre of a nested family of ellipsoids—some having arbitrarily small diameter—all of which are domains of attraction for $z = 0$. This refinement of Thm. B.4 will allow us to conclude that the origin is an asymptotically stable equilibrium point. (See Prop. S.19.)

S.3. Translation. If z is an equilibrium point for F, then the definition $\widetilde{F}(y) := F(z + y)$ leads to an ODE

$$\dot{y} = \widetilde{F}(y) \qquad (S.2)$$

for which $y(\cdot)$ is a solution if and only if $x(\cdot) = z + y(\cdot)$ is a solution of $(S.1)$. In particular, the dynamics of $(S.2)$ around 0 are identical to those of $(S.1)$ around z. From now on, we assume that this transformation has been made and that the equilibrium point of interest lies at $z = 0$.

Flow-invariance and Stability.

S.4. Example (Damped Harmonic Oscillator). Consider the second-order linear equation below, in which $\omega > 0$ and $\zeta \geq 0$ are given constants:

$$\ddot{x} + 2\omega\zeta\dot{x} + \omega^2 x = 0 \iff \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2\omega\zeta x_2 - \omega^2 x_1. \end{cases}$$

Here ω is called the "undamped natural frequency" and ζ is called the "damping ratio." Equations having this form arise in the theory of mechanical vibrations: in this case, they describe the motion of a unit mass under the action of a spring with force constant ω^2 and a frictional force proportional to the velocity. The position of the mass is x_1; its velocity is x_2. At state (x_1, x_2), the total energy of the mass is proportional to

$$V(x_1, x_2) := \omega^2 x_1^2 + x_2^2 \propto \text{potential} + \text{kinetic}.$$

Along any trajectory $(x_1(t), x_2(t))$, the function $v(t) := V(x_1(t), x_2(t))$ records the system's instantaneous energy.

Observe that

$$\begin{aligned} \dot{v}(t) &= \frac{\partial V}{\partial x_1}(x_1(t), x_2(t))\,\dot{x}_1(t) + \frac{\partial V}{\partial x_2}(x_1(t), x_2(t))\,\dot{x}_2(t) \\ &= 2\omega^2 x_1(t)\,[x_2(t)] + 2x_2(t)\left[-\omega^2 x_1(t) - 2\omega\zeta x_2(t)\right] \\ &= -4\omega\zeta x_2(t)^2. \end{aligned}$$

This is no surprise: the energy is constant in the frictionless case $\zeta = 0$ and strictly decreasing when $\zeta > 0$ and $x_2(t) \neq 0$. Therefore the trajectory $(x_1(t), x_2(t))$ is trapped inside the set

$$M := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,:\, V(x, y) \leq V(x_1(0), x_2(0)) \right\}.$$

This is an ellipse centred at the origin, which fits neatly inside the closed ball with centre $(0, 0)$ and radius $\max\{k, k/\omega\}$, where $k^2 = V(x_1(0), x_2(0))$. See the Figure.

Figure. The set $M := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,:\, V(x, y) \leq k^2 \right\}$ when $\omega > 1$.

In particular, the origin is stable: given $\varepsilon > 0$, simply choose $k > 0$ sufficiently small that $\varepsilon > \max\{k, k/\omega\}$; then the set M pictured above is a flow-invariant set for F satisfying both $(0,0) \in \operatorname{int} M$ and $M \subseteq \mathbb{B}(0;\varepsilon)$. ////

Liapunov applied the idea of decreasing energy along trajectories of a physical system to ordinary differential equations of all sorts. The key to his theory is a scalar function which is nonincreasing along trajectories of the system.

S.5. Definition. Let $D \subseteq G$ be any set. A scalar-valued function $V\colon D \to \mathbb{R}$ is a *Liapunov function for F on D* if V is continuously differentiable on a neighbourhood of each point in D, and

$$V(x) \geq 0, \quad \nabla V(x) \bullet F(x) \leq 0 \qquad \forall x \in D.$$

S.6. Theorem (Liapunov). *Let $G \subseteq \mathbb{R}^n$ be an open set, and let $F\colon G \to \mathbb{R}^n$ be C^1. Suppose there exist a continuous function $V\colon G \to \mathbb{R}$ and a constant ℓ such that*

(i) *the set $\Omega_\ell := \{x \in G \,:\, V(x) < \ell\}$ is nonempty and bounded;*

(ii) *for each $\ell' \in (0,\ell)$, one has $\operatorname{cl}\Omega_{\ell'} \subseteq G$;*

(iii) *V is a Liapunov function for F on Ω_ℓ.*

Then Ω_ℓ is a flow-invariant set for F.

Proof. Pick any $\xi \in \Omega_\ell$ and let $x(\cdot;\xi)$ solve $(S.1)$ in Ω_ℓ. Then there is an interval $I(\xi) = (a(\xi), b(\xi))$ containing 0 such that (among other things)

(a) $\dot{x}(t;\xi) = F(x(t;\xi))$ and $x(t;\xi) \in \Omega_\ell$ for each $t \in I(\xi)$; and

(b) either $b(\xi) = +\infty$ or else $\displaystyle\lim_{t\to b(\xi)^-} x(t;\xi)$ is a well-defined point of $\operatorname{bdy}\Omega_\ell$.

It suffices to show that $b(\xi) = +\infty$.

For each $t \in I(\xi)$, the definition of a Liapunov function gives

$$\frac{d}{dt}V(x(t;\xi)) = \nabla V(x(t;\xi))\dot{x}(t;\xi) = \nabla V(x(t;\xi))F(x(t;\xi)) \leq 0.$$

Hence the function $t \mapsto V(x(t;\xi))$ is weakly decreasing on $[0, b(\xi))$, and this implies

$$V(x(t;\xi)) \leq V(x(0;\xi)) = V(\xi) < \ell, \qquad \forall t \in [0, b(\xi)).$$

In particular, let $\ell' = \frac{1}{2}(V(\xi) + \ell)$: this guarantees $\ell' \in (V(\xi), \ell)$, so the previous inequality shows $x(t;\xi) \in \Omega_{\ell'}$ for all $t \in [0, b(\xi))$. But $\operatorname{cl}\Omega_{\ell'}$ is a compact subset of the open set Ω_ℓ, so there is some constant $r > 0$ such that every point in $\operatorname{cl}\Omega_{\ell'}$ lies at a distance no smaller than r from $\partial\Omega_\ell$. In particular, $x(t;\xi)$ cannot approach $\operatorname{bdy}\Omega_\ell$ as t increases, so statement (b) can only hold with $b(\xi) = +\infty$, as required. ////

S.7. Corollary. *In addition to the assumptions of Thm. S.6, suppose $F(\mathbf{0}) = \mathbf{0}$ and*

$$V(x) > 0 = V(\mathbf{0}) \text{ for all } x \in \Omega_\ell,\ x \neq \mathbf{0}.$$

Then $\mathbf{0}$ *is a stable equilibrium point for* F.

Proof. Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$ with $1/n < \ell$, the set $\Omega_{1/n} := \{x \in \Omega : V(x) < 1/n\}$ lies inside Ω_ℓ and is flow-invariant for F, by Thm. S.6. Let us show that $\Omega_{1/n} \subseteq \mathbb{B}[0;\varepsilon]$ for some $n \in \mathbb{N}$. To build a proof by contradiction, suppose not. Then each set $\Omega_{1/n}$ contains some point x_n with $|x_n| \geq \varepsilon$. Now the sequence (x_n) eventually lies in Ω_ℓ, a bounded set, so some subsequence converges to a point $\overline{x} \in \Omega_\ell$. (Notice that $\overline{x}$ lies in each set $\operatorname{cl}\Omega_{1/n}$, $n \in \mathbb{N}$, and all of these closed sets are subsets of Ω_ℓ when $1/n < \ell$.) Now the limit point obeys $|\overline{x}| \geq \varepsilon$, but $V(\overline{x}) = \lim V(x_n) = 0$. This contradicts the assumption that 0 is the unique minimizer of V in the set Ω_ℓ. ////

S.8. Recipe. To use S.6–S.7, choose V first and then adjust ℓ and G. Follow these steps.

1. Choose a likely candidate $V\colon \mathbb{R}^n \to \mathbb{R}$ (some suggestions are given below).
2. Evaluate (or estimate) the set $D = \{x \in \operatorname{dom} F : \nabla V(x)F(x) \leq 0\}$.
3. Consider the sublevel sets $\Omega_\ell^0 := \{x \in \operatorname{dom} V : V(x) < \ell\}$: choose any ℓ for which Ω_ℓ^0 is nonempty, and has a connected component ("piece") that forms a bounded subset of D. Name that connected component G. Then $\Omega_\ell = G$ and V satisfy the hypotheses of Thm. S.6, so Ω_ℓ is a flow-invariant set. If $F(0) = 0$ and Cor S.7 applies, then the origin is a stable equilibrium point.

The reader is encouraged to compare the steps in this recipe with the discussion of Example S.4. The fine points of Step 3 may be easier to understand after reading Example S.9 below.

Choosing a Liapunov function. The applicability of Corollary S.7 rests on the user providing a suitable function V. There is plenty of room for art and inspiration here, but some general suggestions can be given.

(a) Exploit some physical analogy: For a system like $\ddot{x} + R(x,\dot{x})\dot{x} + \phi(x) = 0$, rendered in first-order form by substituting $x_1 = x$, $x_2 = \dot{x}$, guess the energy-like function

$$V(x_1,x_2) := \tfrac{1}{2}x_2^2 + \int_0^{x_1} \phi(r)\,dr.$$

This will usually work if $R(x_1,x_2) \geq 0$ for (x_1,x_2) near $(0,0)$. See Example S.4 above.

(b) Manipulate of the basic equations: Try to find first integrals or quantities which decrease along the solution curves. For example, since $\frac{d}{dt}\left(x_k(t)^2\right) = 2x_k(t)\dot{x}_k(t)$, multiplying both sides of a given differential equation like $\dot{x}_k = \cdots$ by $2x_k(t)$ will produce a right-hand side that predicts part of what you would see in $\nabla V \bullet F$ if V included a term of the form x_k^2. See Example S.9 below.

(c) Try a positive quadratic form: Guess $V(x) = x^T K x$ for some symmetric positive definite matrix K, and then use the conditions of Thm. S.6 to help select the entries of K. See Example S.10 below, and the subsequent theoretical developments.

S.9. Example. *Prove that $x = 0$, $\dot{x} = 0$ is a stable equilibrium point for the scalar system below, in which the given quantities R, L, C, α, are positive constants:*

$$L\ddot{x} - \alpha\dot{x}^2\ddot{x} + R\dot{x} + \frac{1}{C}x = 0.$$

Describe a system of flow-invariant subsets of the phase plane, each containing the origin. (This equation describes the electrical charge x on the capacitor in a series RLC circuit where the voltage drop across the inductor is not simply $\frac{d}{dt}(L\dot{x})$, but the more precise nonlinear expression $\frac{d}{dt}\left(L\dot{x} - \frac{\alpha}{3}\dot{x}^3\right)$.)

Solution. A linear change of time scale will simplify the equations. Let $y(t) := x(t/\omega)$ to get $\omega\dot{y}(t) = \dot{x}(t/\omega)$, $\omega^2\ddot{y}(t) = \ddot{x}(t/\omega)$, and

$$\omega^2 L\ddot{y} - \alpha\omega^4\dot{y}^2\ddot{y} + R\omega\dot{y} + \frac{1}{C}y = 0.$$

The inspired choice $\omega^2 = 1/\sqrt{LC}$ leads to the equation

$$\ddot{y} - \beta\dot{y}^2\ddot{y} + \mu\dot{y} + y = 0,$$

in which $\beta = \alpha/(L^2C) > 0$ accounts for the nonlinearity and $\mu = R(C/L)^{1/2} \geq 0$ relates the damping effects of the resistor R. Define $x_1 = y$, $x_2 = \dot{y}$ to get the first-order system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{x_1 + \mu x_2}{1 - \beta x_2^2}. \end{aligned}$$

If we clear the denominators of the second equation, the system becomes

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2\left(1 - \beta x_2^2\right) &= -x_1 - \mu x_2. \end{aligned}$$

Multiply the first equation by x_1 and the second by x_2 and add: the result is

$$\frac{d}{dt}\left[\tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 - \tfrac{1}{4}\beta x_2^4\right] = -\mu x_2^2.$$

This computation reveals that for the function

$$V(x_1, x_2) := x_2^2 + x_1^2 - \tfrac{1}{2}\beta x_2^4,$$

one has

$$\nabla V(x) \bullet F(x) = -2\mu x_2^2 \leq 0 \qquad \forall x \in \mathbb{R}^2.$$

In the terminology of Recipe S.8, $D = \mathbb{R}^2$. We consider the level sets of V: for each $\ell \in \mathbb{R}$, the set $\Omega_\ell^0 = \{x \in \mathbb{R}^2 : V(x) < \ell\}$ is unbounded, but for sufficiently small $\ell > 0$, these sets contain a bounded component centred at the origin. Completing the square reveals

$$V(x) = x_1^2 - \frac{\beta}{2}\left(x_2^2 - \frac{1}{\beta}\right)^2 + \frac{1}{2\beta}.$$

The contours of constant V in the (x_1, x_2)-plane are intricate curves, but one is easy:

$$V(x) = \frac{1}{2\beta} \iff x_1 = \pm\left(\frac{\beta}{2}x_2^2 - \frac{1}{2}\right).$$

This reveals $\ell = 1/(2\beta)$ as the largest value for which the set Ω_ℓ^0 has a bounded connected component containing the origin. We choose this component for G:

$$G = \left\{(x_1, x_2) \,:\, \frac{\beta}{2}x_2^2 - \frac{1}{2} < x_1 < \frac{1}{2} - \frac{\beta}{2}x_2^2\right\}.$$

With this choice of G, each set $\Omega'_\ell = \{x \in G : V(x) < \ell'\}$ for $0 \le \ell' \le \ell = 1/(2\beta)$ is a bounded subset of G. Theorem S.6 applies to show that each of these sets is flow-invariant for F; Corollary S.7 guarantees that the origin is indeed a stable equilibrium point for F. ////

S.10. Example. *Use a suitable quadratic Liapunov function to prove that the origin is a stable equilibrium point for the system shown below:*

$$\begin{aligned} \dot{x}_1 &= -4x_1^3 + x_2^3, \\ \dot{x}_2 &= -4x_1x_2^2. \end{aligned}$$

Solution. Consider $V(x_1, x_2) := \alpha x_1^2 + \gamma x_2^2$, where α and γ are positive real numbers to be chosen later. For this function,

$$\begin{aligned} \nabla V(x) \bullet F(x) &= \langle 2\alpha x_1,\, 2\gamma x_2\rangle \bullet \big\langle -4x_1^3 + x_2^3,\, -4x_1x_2^2\big\rangle \\ &= -8\alpha x_1^4 + (2\alpha - 8\gamma)x_1x_2^3. \end{aligned}$$

Choose $\gamma = 1$, $\alpha = 4$ to get rid of the term which is not sign-definite: this gives

$$\nabla V(x) \bullet F(x) = -32x_1^4 \le 0,$$

so $D = \mathbb{R}^2$. For any $\ell > 0$, we may choose $G = \Omega_\ell$, a bounded ellipse symmetric with respect to both coordinate axes. Every such set is open and contains the origin, so it's flow-invariant by Thm. S.6 and the origin is stable by Corollary S.7. ////

Note that in more general problems, it may be necessary to guess a quadratic Liapunov function involving mixed terms. When $n = 2$, such functions have the form

$$V(x_1, x_2) = \alpha x_1^2 + 2\beta x_1x_2 + \gamma x_2^2.$$

This V obeys the hypotheses of Cor. S.7 if and only if $\alpha > 0$ and $\alpha\gamma - \beta^2 > 0$.

Domains of Attraction and Asymptotic Stability. To pass from the elementary theory of stability to the more delicate question of asymptotic stability requires further analysis, some quite beautiful and complex. Stability describes the situation in which disturbances of a given system do not grow as time progresses; asymptotic stability requires that they actually decay.

Consider the situation described in Theorem S.6, where an F-trajectory evolves in a bounded set Ω_ℓ in such a way that $v(t) \stackrel{\text{def}}{=} V(x(t;\xi))$ is weakly decreasing and bounded below. This guarantees the existence of the limit

$$\overline{v} \stackrel{\text{def}}{=} \lim_{t\to\infty} V(x(t;\xi)).$$

We can exploit this as follows. Clearly

$$0 = \overline{v} - \overline{v} = \lim_{n\to\infty} \left[V(x(n+1;\xi)) - V(x(n;\xi)) \right].$$

So fix n and applying the Mean Value Theorem to v on $[n, n+1]$: it shows that some $\theta_n \in (n, n+1)$ must obey

$$0 = \lim_{n\to\infty} \nabla V(x(\theta_n;\xi)) \bullet F(x(\theta_n)).$$

Since the sequence $x(\theta_n)$ is bounded and finite-dimensional, it must have a subsequence converging to some point $\overline{x}$ in Ω_ℓ: along this subsequence, continuity gives

$$0 = \nabla V(\overline{x}) \bullet F(\overline{x}).$$

So under the conditions of Theorem S.6, every F-trajectory starting in Ω_ℓ has the property that as $t \to \infty$, the state $x(t)$ repeatedly (at times θ_n) comes very close to the set defined by

$$D_0 = \{x \in \Omega_\ell \,:\, \nabla V(x) \bullet F(x) = 0\}.$$

If $D_0 = \{0\}$, this implies that the origin is asymptotically stable. To see why, fix any $\varepsilon > 0$ and note that for some $\lambda \in (0, \ell)$ one has $\Omega_\lambda \subseteq \mathbb{B}[0;\varepsilon]$. Build a sequence θ_n as shown above and note that since $0 \in \operatorname{int} \Omega_\lambda$, we will have $x(\theta_n) \in \Omega_\lambda$ for some large n. But flow invariance then gives $x(t) \in \Omega_\lambda \subseteq \mathbb{B}[0;\varepsilon]$ for all $t \ge \theta_n$, as required. ////

What if the set D_0 is not just a single point? Here the story gets more complicated, and we reach for some heavier theory.

S.11. Proposition. *Let $G \subseteq \mathbb{R}^n$ be an open set on which a continuously differentiable function $F\colon G \to \mathbb{R}^n$ is defined. Suppose that $\xi \in G$ is an initial point for which the solution $x(\cdot;\xi)$ of $(S.1)$ exists for all $t \ge 0$, and that some compact subset C of G contains all the values $x(t;\xi)$, $t \ge 0$. Consider the "ω-limit set" defined by*

$$\omega(\xi) := \{z \in \mathbb{R}^n \,:\, z = \lim x(t_i;\xi) \text{ for some sequence } t_i \to \infty\}.$$

(a) *Under the hypotheses above, $\omega(\xi)$ is a nonempty compact flow-invariant subset of G.*

(b) *If there exists a Liapunov function V for F on the set C, then $\nabla V(z) \bullet F(z) = 0$ for all $z \in \omega(\xi)$.*

Proof. (a) For any sequence $t_i \to \infty$, the corresponding sequence $x_i := x(t_i; \xi)$ is contained in the compact set C. Therefore it has a convergent subsequence, and the limit of this subsequence (which certainly lies in C) is a point of $\omega(\xi)$. Hence $\omega(\xi)$ is a nonempty subset of C.

To see that $\omega(\xi)$ is closed, suppose $\{z_k\}$ is a sequence in $\omega(\xi)$ converging to some point z_∞. For each k, we have

$$z_k = \lim_{i\to\infty} x(t_i^{(k)}; \xi) \text{ for some sequence } t_i^{(k)} \xrightarrow[i\to\infty]{} \infty.$$

So for each k, choose $i = i(k)$ so big that $t_{i(k)}^{(k)} > k$ and

$$\left| x\left(t_{i(k)}^{(k)}; \xi\right) - z_k \right| < 1/k.$$

For the sequence $s_k := t_{i(k)}^{(k)}$, we have $s_k > k$ for all k, so $s_k \to \infty$, while

$$x(s_k; \xi) = z_\infty + (z_k - z_\infty) + (x(s_k; \xi) - z_k).$$

Thus $z_\infty = \lim x(s_k; \xi)$, i.e., $z_\infty \in \omega(\xi)$ as required.

Finally, we show that $\omega(\xi)$ is flow-invariant. To do so, let $z \in \omega(\xi)$. This means that

$$z = \lim x(t_i; \xi) \text{ for some sequence } t_i \to \infty,$$

so we may define $x_i := x(t_i; \xi)$. Now for any $\bar{t}$ in the interval of existence $I(z)$ provided by Thm. A.1, the function $x(\bar{t}; \cdot)$ is continuous at z, so

$$\begin{aligned} x(\bar{t}; z) &= \lim x(\bar{t}; x_i) \\ &= \lim x(\bar{t}; x(t_i; \xi)) \\ &= \lim x(t_i + \bar{t}; \xi). \end{aligned}$$

This shows that $x(\bar{t}; z) \in \omega(\xi)$. Since this argument applies to any $\bar{t} \in I(z)$, it follows that $b(z) = +\infty$ (Thm. A.1(d)) and hence that $\omega(\xi)$ is a flow-invariant set for F.

(b) Now suppose that V is a Liapunov function for F on the set C. Then the function $t \mapsto V(x(t; \xi))$ (which is defined on $[0, +\infty)$) is nonincreasing, since its derivative is nonpositive:

$$\frac{d}{dt} V(x(t; \xi)) = \nabla V(x(t; \xi)) \bullet F(x(t; \xi)) \le 0 \qquad \forall t \ge 0.$$

Since V is continuous and C is compact, this function is bounded below by the finite number $\min_{x\in C} V(x)$. Hence it makes sense to define the constant

$$\overline{V} := \lim_{t\to\infty} V(x(t;\xi)).$$

For any $z \in \omega(\xi)$ one has $z = \lim x(t_i;\xi)$ for some sequence $t_i \to \infty$. Hence

$$V(z) = V(\lim x(t_i;\xi)) = \lim V(x(t_i;\xi)) = \overline{V}.$$

Thus $V(z) = \overline{V}$ for all $z \in \omega(\xi)$. But $\omega(\xi)$ is flow-invariant. So for any $z \in \omega(\xi)$ we have $V(x(t;z)) = \overline{V}$ for all $t \geq 0$. This implies that

$$\forall t > 0, \quad 0 = \frac{d}{dt}V(x(t;z)) = \nabla V(x(t;z)) \bullet F(x(t;z)).$$

In the limit as $t \to 0^+$, we find that

$$0 = \nabla V(x(0;z)) \bullet F(x(0;z)) = \nabla V(z) \bullet F(z),$$

as required. ////

Here is a refinement of Theorem S.6.

S.12. Theorem (Liapunov/LaSalle). *Let $G \subseteq \mathbb{R}^n$ be an open set, on which a continuously differentiable mapping $F\colon G \to \mathbb{R}^n$ is given. Suppose there exist a continuous function $V\colon G \to \mathbb{R}$ and a constant ℓ such that*

(i) the set $\Omega_\ell := \{x \in G : V(x) < \ell\}$ is nonempty and bounded, with $\mathrm{cl}\,\Omega_\ell \subseteq G$;

(ii) V is a Liapunov function for F on Ω_ℓ.

Then Ω_ℓ is a flow-invariant set for F. Moreover, the set

$$D_0 := \{x \in \Omega_\ell : \nabla V(x)F(x) = 0\}$$

contains a flow-invariant subset M of Ω_ℓ with the following property: for every $\xi \in \Omega_\ell$, one has $x(t;\xi) \to M$ as $t \to \infty$. (The last assertion means that the scalar quantity $\inf\{|x(t;\xi) - m| : m \in M\}$ tends to 0 as $t \to \infty$.)

Proof. The flow-invariance of Ω_ℓ follows immediately from Thm. S.6. To prove the additional conclusions, define

$$M = \bigcup_{\xi\in\Omega} \omega(\xi).$$

Notice that for each ξ in Ω_ℓ, the function V is constant on $\omega(\xi)$ by the proof of Prop. S.11, and the constant value of V there is at most $V(\xi) < \ell$. Consequently $\omega(\xi)$ is a subset of Ω_ℓ, and $\omega(\xi) \subseteq D_0$ by Prop. S.11(b). This shows that M, as the union of flow-invariant sets subsets of D_0, is itself a flow-invariant subset of D_0. And for any $\xi \in \Omega_\ell$, we have

$$\min\{|x(t;\xi) - m| \,:\, m \in M\} \leq \min\{|x(t;\xi) - m| \,:\, m \in \omega(\xi)\}.$$

We claim that the RHS converges to 0 as $t \to \infty$. To prove this, suppose not. Then there would be some $\varepsilon > 0$ and some sequence $t_i \to \infty$ such that

$$|x(t_i;\xi) - m| \geq \varepsilon \quad \forall m \in \omega(\xi),\ \forall i. \tag{$**$}$$

But the sequence $x(t_i;\xi)$ is bounded (since it lies in Ω_ℓ), so it has a convergent subsequence. The limit point of this subsequence lies in $\omega(\xi)$ by definition, contradicting $(**)$. ////

When seeking to confirm the asymptotic stability of an equilibrium point at the origin, the following simple extension of Thm. S.12 is useful.

S.13. Corollary. *Suppose $F(0) = 0$. If the conditions of Thm. S.12 can be satisfied by some Liapunov function V and constant ℓ with the additional properties that $0 \in \Omega_\ell$ and*

$$V(x) > V(0) \text{ for all } x \in \Omega_\ell,\ x \neq 0,$$

then 0 is a stable equilibrium point for F. If, in addition, the only F-trajectory contained in D_0 is $x(t) \equiv 0$, then 0 is asymptotically stable, and Ω_ℓ is a domain of attraction.

Proof. The first conclusion was derived in Cor. S.7.

To get the second, note that Thm. S.12 implies that every set Ω_λ for $V(0) < \lambda < \ell$ is a domain of attraction for $z = 0$. The arguments of the Cor. S.7 show that by reducing $\lambda > V(0)$, the diameters of the sets Ω_λ can be made arbitrarily small without compromising the assertion that $0 \in \operatorname{int}\Omega_\lambda$. ////

S.14. Recipe. Follow the same steps as in Recipe S.8, adding the following:

4. Compute $D_0 = \{x \in \Omega_\ell \,:\, \nabla V(x) F(x) = 0\}$ and let M be its largest flow-invariant subset. (This is precisely the set M constructed in the proof above; note that $0 \in M$.) Then every trajectory for F which starts in Ω converges to M as $t \to \infty$.

S.15. Example. Recall example S.4, where we had

$$F(x_1, x_2) = \begin{bmatrix} x_2 \\ -2\omega\zeta x_2 - \omega^2 x_1 \end{bmatrix}, \quad V(x_1, x_2) = \omega^2 x_1^2 + x_2^2.$$

The set D in Step 2 of Recipe S.8 is $\mathbb{R}^2$, and for every $\ell > 0$ the set $\Omega_\ell^0 = \{x \in \mathbb{R}^2 \,:\, V(x) < \ell\}$ is a bounded open ellipse in $\mathbb{R}^2$. Since each of these sets is already bounded, we can simply take $G = \mathbb{R}^2$ (the natural domain of F) and $\Omega_\ell = \Omega_\ell^0$ for any $\ell > 0$. Then

$$D_0 = \begin{cases} \Omega_\ell, & \text{if } \zeta = 0 \text{ (no friction)},\\ \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,:\, x_2 = 0 \right\}, & \text{if } \zeta > 0 \text{ (some friction)}. \end{cases}$$

If $\zeta = 0$, Thm. S.6 asserts that each ellipse Ω_ℓ is a flow-invariant set. We have $D_0 = \Omega_\ell$, so it is impossible to get a more precise conclusion using Thm. S.12.
If $\zeta > 0$, we can estimate M, by looking for those initial points ξ in D_0 from which the trajectory stays in D_0 forever. Now if $x_2(t) = 0$ for all $t \geq 0$ along some F-trajectory, then $\dot{x}_2(t) = 0$ for all $t \geq 0$ also, so $\omega^2 x_1(t) = -2\omega\zeta x_2(t) - \dot{x}_2(t) = 0$ for all $t \geq 0$. Hence $x_1(t) = 0$ for all $t \geq 0$. This shows that the only F-trajectory inside D_0 is the constant function $(x_1(t), x_2(t)) = (0, 0)$. Thus $M = \{(0, 0)\}$. Theorem S.12 now asserts that all solutions starting in Ω_ℓ converge to the origin as $t \to \infty$. Since the ellipsoid Ω_ℓ can be made arbitrarily large by a suitable choice of $\ell > 0$, it follows that any solution of the differential equation converges to $(0, 0)$ as $t \to \infty$. (Of course this is a wellk-known fact, but it provides a nice familiar context to see Liapunov's methods at work.) ////

S.16. Example S.9 continued. Under the assumption that $\mu > 0$, we see that D_0 consists of the points in Ω_ℓ where $x_2 = 0$: the only F-trajectory contained in D_0 is the zero trajectory, so Cor. S.13 asserts that $(0, 0)$ is an asymptotically stable equilibrium point for F. Theorem S.12 ensures that any trajectory starting in Ω_ℓ converges to the origin as $t \to \infty$.

S.17. Example S.10 continued. The set D_0 in this example is that part of Ω_ℓ lying along the x_2-axis, and it is easy to show that $M = \{(0, 0)\}$. By Corollary S.13, the origin is an asymptotically stable equilibrium point for the given system; indeed, Thm. S.7 implies that any trajectory starting in Ω_ℓ converges to the origin. Since arbitrarily large choices of $\ell > 0$ are consistent with the arguments above, it follows that the solution to this equation starting from any initial point in $\mathbb{R}^2$ converges to the origin. ////

Liapunov Functions as Theoretical Tools. Often a quadratic form $V(x) := x^T K x$ makes a serviceable Liapunov function. In order that the level sets of V be bounded, we must assume that the $n \times n$ matrix K is positive definite; no generality is lost in assuming that K is symmetric. Our next result shows that a Liapunov function of this form can always be used when the differential equation is linear and all the eigenvalues of its coefficient matrix has eigenvalues have negative real parts. To understand the origin of condition (b) in the following statement, let A be an $n \times n$ matrix and $F(x) := Ax$. Then for $V(x) := x^T K x$, we have

$$\nabla V(x) F(x) = 2x^T K A x = x^T \left(KA + A^T K \right) x.$$

In order that this quantity be nonpositive throughout some neighbourhood of the origin, it is necessary and sufficient that the matrix K be chosen so that $KA + A^T K$ is negative semidefinite.

S.18. *Theorem. Let $A \in \mathbb{R}^{n\times n}$. The following assertions are equivalent.*

(a) $\Re e(\lambda) < 0$ *for all* $\lambda \in \sigma(A)$*;*

(b) for every symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$, the matrix equation

$$-Q = KA + A^T K$$

has a symmetric positive definite solution $K \in \mathbb{R}^{n\times n}$. (This equation is called "the Liapunov equation.")

Proof. (a⇒b) Given a symmetric matrix $Q > 0$, define

$$K = \int_0^\infty e^{A^T t} Q e^{At}\, dt.$$

This integral converges because all the entries in the matrices e^{At} and $e^{A^T t}$ tend to zero exponentially fast, as shown in Section B. Note also that $K = K^T$. Moreover,

$$\begin{aligned} A^T K &= \int_0^\infty A^T e^{A^T t} Q e^{At}\, dt \\ &= \int_0^\infty \left(\frac{d}{dt} e^{A^T t}\right) Q e^{At}\, dr \\ &= e^{A^T t} Q e^{At}\Big|_0^\infty - \int_0^\infty e^{A^T t} Q e^{At} A\, dt \\ &= -Q - KA. \end{aligned}$$

Hence $A^T K + KA = -Q$, as required.

(b⇒a) If (b) holds, choose $Q = I$ and let $K = K^T > 0$ obey $A^T K + KA = -I$. Then define $V(x) := x^T K x$. Observe that for $F(x) := Ax$,

$$\nabla V(x) F(x) = 2x^T KAx = x^T \left(KA + A^T K\right) x = -|x|^2.$$

By Theorem S.6, each of the bounded ellipsoids $\Omega_\ell = \{x \in \mathbb{R}^n : V(x) < \ell\}$ $(\ell > 0)$ is a flow-invariant set. In the notation of Recipe S.8, we have $D_0 = \{0\}$ for any such ellipsoid, so $x(t;\xi) \to 0$ for every $\xi \in \Omega_\ell$. According to Thm. B.4, assertion (a) follows. ////

Note that the proof of Thm. S.18 shows that if all the eigenvalues of a matrix A have negative real parts, then the origin is an asymptotically stable equilibrium point for the system $\dot{x} = Ax$. This is a consequence of Cor. S.13.

In fact, Thm. S.18 implies a stronger result: if a nonlinear system has an asymptotically stable linearization about some equilibrium point, then that equilibrium point is asymptotically stable for the full nonlinear system.

S.19. Proposition. *Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be smooth and obey $F(0) = 0$. Define $A = DF(0)$, the $n \times n$ Jacobian matrix of F at 0. If every eigenvalue λ of A obeys $\Re e(\lambda) < 0$, then 0 is an asymptotically stable equilibrium point for the nonlinear system*

$$\dot{x} = F(x).$$

Proof. Since all the eigenvalues of A have negative real parts, Thm. S.18 asserts that some symmetric matrix $K > 0$ satisfies $A^T K + KA = -I$. We use this matrix to define $V(x) := x^T K x$. Then we have

$$\begin{aligned}
\nabla V(x) F(x) &= 2x^T K F(x) \\
&= 2x^T K A x + 2x^T K(F(x) - Ax) \\
&= x^T (KA + A^T K)x + 2x^T K(F(x) - Ax) \\
&= -|x|^2 + 2x^T K(F(x) - Ax).
\end{aligned}$$

Now by definition $A = DF(0)$ means that

$$\lim_{x \to 0} \frac{F(x) - F(0) - Ax}{|x|} = 0,$$

so in particular there exists $\delta > 0$ so small that

$$\frac{|F(x) - Ax|}{|x|} \leq \frac{1}{4\|K\|} \qquad \forall x \in \mathbb{B}(0;\delta). \tag{$*$}$$

Hence whenever $|x| < \delta$, we have

$$\left|2x^T K(F(x) - Ax)\right| \leq 2|x|\|K\| \frac{|x|}{4\|K\|} = \tfrac{1}{2}|x|^2,$$

which implies

$$\nabla V(x) F(x) \leq -|x|^2 + \tfrac{1}{2}|x|^2 = -\tfrac{1}{2}|x|^2 \leq 0 \qquad \forall x \in \mathbb{B}(0;\delta).$$

Upon choosing $\ell > 0$ so small that

$$\Omega_\ell = \{x \in \mathbb{R}^n \,:\, V(x) < \ell\} \subseteq \mathbb{B}(0;\delta),$$

we may apply Thm. S.12: that result asserts that Ω_ℓ is flow-invariant, and that for every $\xi \in \Omega_\ell$ one has $x(t;\xi) \to 0$ as $t \to \infty$. In other words, Ω_ℓ is a domain of attraction for $z = 0$. Clearly $0 \in \operatorname{int} \Omega_\ell$ for any $\ell > 0$, while for any $\varepsilon > 0$ we can arrange $\Omega_\ell \subseteq \mathbb{B}(0;\varepsilon)$ simply by choosing $\ell > 0$ sufficiently small. This confirms the definition of asymptotic stability. ////

Proposition S.18 allows us to assess the stability of a nonlinear system at an equilibrium point by looking at its linearization: if the linearization involves a matrix whose eigenvalues all have negative real parts, then the equilibrium point in question is asymptotically stable for the nonlinear system. To illustrate, recall Example S.9, where

$$F(x) = \begin{bmatrix} x_2 \\ -\dfrac{x_1 + \lambda x_2}{1 - \beta x_2^2} \end{bmatrix}.$$

For this nonlinear function F, we have $F(0) = 0$ and

$$DF(0) = \begin{bmatrix} 0 & 1 \\ -1 & -\lambda \end{bmatrix}.$$

Both the eigenvalues of this matrix have negative real parts whenever $\lambda > 0$, so Prop. S.19 implies that the origin is an asymptotically stable equilibrium point for F. Prop. S.19 has the advantage that it is not necessary to find a Liapunov function V to deduce stability; the disadvantage of this result is that it gives no quantitative estimate of the region about the origin from which all trajectories converge to zero. However, the proof does show that some quadratic Liapunov function $V(x) = x^T Kx$, chosen so that $DF(0)^T K + KDF(0) = -I$, will at least provide local information about a domain of attraction for the nonlinear system.

The previous two results have shown that a quadratic Liapunov function can always be used to detect asymptotic stability for systems whose linearization is stable. Of course, quadratic functions may be used in other cases, too: in Example S.10, we had

$$F(x) = \begin{bmatrix} -4x_1^3 + x_2^3 \\ -4x_1 x_2^2 \end{bmatrix}, \quad \text{and} \quad DF(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

but a quadratic V still served very well.

A result complementary to Prop. S.19 can also be established: if $F(0) = 0$ and $A = DF(0)$ has an eigenvalue whose real part is positive, then the origin is definitely not a stable equilibrium point for $\dot{x} = F(x)$.