M403(2012) Solutions—Problem Set 1

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1. The number $\lambda \in \mathbb{C}$ is an eigenvalue for A exactly when

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & -1 \\ -k & -1-\lambda \end{bmatrix} = (-1-\lambda)^2 - k,$$

i.e., when $\lambda = -1 \pm \sqrt{k}$.

(i) When $k = \frac{1}{2}$, the eigenvalues are $-1 \pm \sqrt{\frac{1}{2}}$, or approximately -1.7071 and -0.2929. Corresponding eigenvectors $\mathbf{u}$ and $\mathbf{v}$ are given by

$$\lambda = -1 + \sqrt{\tfrac{1}{2}} \implies -\sqrt{\tfrac{1}{2}}u_1 - u_2 = 0 \implies \mathbf{u} = (-\sqrt{2},\, 1),$$
$$\lambda = -1 - \sqrt{\tfrac{1}{2}} \implies \sqrt{\tfrac{1}{2}}v_1 - v_2 = 0 \implies \mathbf{v} = (\sqrt{2},\, 1),$$

The general solution of $(*)$ in this case is

$$\mathbf{x}(t) = C_1 e^{(-1+1/\sqrt{2})t} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} + C_2 e^{(-1-1/\sqrt{2})t} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \qquad C_1, C_2 \in \mathbb{R}. \tag{1}$$

(ii) When $k = 2$, the eigenvalues are $-1 \pm \sqrt{2}$, or approximately 0.4142 and -2.4142. Corresponding eigenvectors $\mathbf{u}$ and $\mathbf{v}$ are given by

$$\lambda = -1 + \sqrt{2} \implies -\sqrt{2}u_1 - u_2 = 0 \implies \mathbf{u} = (1,\, -\sqrt{2}),$$
$$\lambda = -1 - \sqrt{2} \implies \sqrt{2}v_1 - v_2 = 0 \implies \mathbf{v} = (1,\, \sqrt{2}),$$

The general solution of $(*)$ in this case is

$$\mathbf{x}(t) = C_1 e^{(-1+\sqrt{2})t} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + C_2 e^{(-1-\sqrt{2})t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \qquad C_1, C_2 \in \mathbb{R}. \tag{2}$$

(iii) When $k = \frac{1}{2}$, both eigenvalues of A are negative, so each trajectory in (1) converges to 0 as $t \to \infty$. By contrast, when $k = 2$, the first eigenvalue of A is positive. Hence each trajectory in (2) with $C_1 \neq 0$ obeys $|\mathbf{x}(t)| \to \infty$ as $t \to \infty$. The general formula for the eigenvalues, $\lambda = -1 \pm \sqrt{k}$, shows that transition between these situations happens when $k = 1$: for $k < 1$ both eigenvalues have negative real part (they are complex when $k < 0$), whereas whenever $k > 1$, one eigenvalue will be a positive real number.

2. (i) Here $A = \lambda I + M$, where

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The ODE system $\dot{\mathbf{x}} = M\mathbf{x}$, in components, says

$$\dot{x}_1 = 0, \qquad \dot{x}_2 = x_1, \qquad \dot{x}_3 = 0.$$

Therefore the general solution has $x_1(t) = c_1$ and $x_3(t) = c_3$ for some constants c_1 and c_3, while $x_2(t) = c_1 t + c_2$ for some constant c_2. With this notation, $\mathbf{x}(0) = \mathbf{c}$, and it follows that

$$\mathbf{x}(t; \vec{\xi}) = \vec{\xi} + t\xi_1 \widehat{\mathbf{e}}_2.$$

Thus we obtain

$$e^{Mt} = [\mathbf{x}(t;\widehat{\mathbf{e}}_1) \mid \mathbf{x}(t;\widehat{\mathbf{e}}_2) \mid \mathbf{x}(t;\widehat{\mathbf{e}}_3)] = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, since λI commutes with M, we can write

$$e^{At} = e^{(\lambda I + M)t} = e^{\lambda I t} e^{Mt} = e^{\lambda t} \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ te^{\lambda t} & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

(ii) Expanding the determinant along the top row, we find

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ -1 & 1 & 1-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda)(1-\lambda) + (2-\lambda)$$
$$= (2-\lambda)\left[(3-\lambda)(1-\lambda) + 1\right] = (2-\lambda)^3.$$

Thus A has a triple eigenvalue at 2. Since the matrix $2I$ commutes with any other matrix, the laws of matrix exponents give

$$e^{At} = e^{(A-2I)t + 2It} = e^{2It} e^{(A-2I)t}.$$

Writing $M = A - 2I$, we have

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}.$$

Writing $\dot{\mathbf{x}} = M\mathbf{x}$ in components, we have

$$\dot{x}_1 = x_1 + x_3, \qquad \dot{x}_2 = 0, \qquad \dot{x}_3 = -x_1 + x_2 - x_3.$$

Therefore $x_2(t) = c_2$ (some constant) in general, while

$$\ddot{x}_1 = \dot{x}_1 + \dot{x}_3 = (x_1 + x_3) + (-x_1 + x_2 - x_3) = x_2(t) = c_2.$$

Thus $x_1(t) = \frac{1}{2}c_2 t^2 + kt + c_1$ for some constants c_1 and k, and then the first equation gives

$$x_3 = \dot{x}_1 - x_1 = (c_2 t + k) - (\tfrac{1}{2}c_2 t^2 + kt + c_1) = -\tfrac{1}{2}c_2 t^2 + (c_2 - k)t + (k - c_1).$$

Let's define $c_3 = k - c_1$ to eliminate $k = c_1 + c_3$. This gives $\mathbf{x}(0) = \mathbf{c}$, and hence

$$\mathbf{x}(t;\vec{\xi}) = \begin{bmatrix} \xi_1 + (\xi_1 + \xi_3)t + \frac{1}{2}\xi_2 t^2 \\ \xi_2 \\ \xi_3 + (\xi_2 - \xi_1 - \xi_3)t - \frac{1}{2}\xi_2 t^2 \end{bmatrix} \implies e^{Mt} = [\mathbf{x}(t;\widehat{\mathbf{e}}_1) \mid \mathbf{x}(t;\widehat{\mathbf{e}}_2) \mid \mathbf{x}(t;\widehat{\mathbf{e}}_3)] = \begin{bmatrix} 1+t & t^2/2 & t \\ 0 & 1 & 0 \\ -t & t - t^2/2 & 1-t \end{bmatrix}.$$

Consequently

$$e^{At} = e^{2It} e^{Mt} = e^{2t} \begin{bmatrix} 1+t & t^2/2 & t \\ 0 & 1 & 0 \\ -t & t - t^2/2 & 1-t \end{bmatrix}.$$

Remark. In both parts, a power-series approach also works well: in part (i), one has $M^2 = 0$; in part (ii), $M^3 = 0$.

3. In components, the ODE system $\dot{\mathbf{x}} = A\mathbf{x}$ becomes

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{\omega^2}{R}x_3, \qquad \dot{x}_3 = \frac{1}{R}x_2.$$

The third equation implies $\ddot{x}_3 = \frac{1}{R}\dot{x}_2$, which the second equation reduces to

$$\ddot{x}_3 = \frac{1}{R}\left(-\frac{\omega^2}{R}x_3\right), \qquad \text{i.e.,} \qquad \ddot{x}_3 + \frac{\omega^2}{R^2}x_3 = 0.$$

The general solution is $x_3(t) = \alpha\cos(\omega t/R) + \beta\sin(\omega t/R)$ for arbitrary $\alpha, \beta \in \mathbb{R}$. Now the third equation gives

$$x_2(t) = R\dot{x}_3(t) = -\alpha\omega\sin(\omega t/R) + \beta\omega\cos(\omega t/R),$$

and the first equation gives some constant γ such that

$$\dot{x}_1 = x_2 = -\alpha\omega\sin(\omega t/R) + \beta\omega\cos(\omega t/R), \qquad \text{so} \qquad x_1(t) = \alpha R\cos(\omega t/R) + \beta R\sin\omega t/R) + \gamma.$$

The general solution for $\mathbf{x}$ involves the three parameters α, β, γ:

$$\mathbf{x} = \begin{bmatrix} \alpha R\cos(\omega t/R) + \beta R\sin(\omega t/R) + \gamma \\ -\alpha\omega\sin(\omega t/R) + \beta\omega\cos(\omega t/R) \\ \alpha\cos(\omega t/R) + \beta\sin(\omega t/R) \end{bmatrix}$$

Note that $\mathbf{x}(0) = (\alpha R + \gamma,\ \omega\beta,\ \alpha)$.

Therefore

$$\begin{aligned}
\mathbf{x}(0) = \mathbf{e}_1 &\iff \alpha = 0,\ \beta = 0,\ \gamma = 1 &&\iff \mathbf{x}(t) = (1,\ 0,\ 0)\,, \\
\mathbf{x}(0) = \mathbf{e}_2 &\iff \alpha = 0,\ \beta = 1/\omega,\ \gamma = 0 &&\iff \mathbf{x}(t) = ((R/\omega)\sin(\omega t/R),\ \cos(\omega t/R),\ (1/\omega)\sin(\omega t/R))\,, \\
\mathbf{x}(0) = \mathbf{e}_3 &\iff \alpha = 1,\ \beta = 0,\ \gamma = -R &&\iff \mathbf{x}(t) = (R[\cos(\omega t/R) - 1],\ -\omega\sin(\omega t/R),\ \cos(\omega t/R))\,,
\end{aligned}$$

These provide the columns of the matrix e^{At}:

$$e^{At} = \begin{bmatrix} 1 & (R/\omega)\sin(\omega t/R) & R\cos(\omega t/R) - R \\ 0 & \cos(\omega t/R) & -\omega\sin(\omega t/R) \\ 0 & (1/\omega)\sin(\omega t/R) & \cos(\omega t/R) \end{bmatrix}$$

This reduces correctly to I when $t = 0$, and the comparison below confirms that the calculation is correct:

$$\frac{d}{dt}\begin{bmatrix} 1 & (R/\omega)\sin(\omega t/R) & R\cos(\omega t/R) - R \\ 0 & \cos(\omega t/R) & -\omega\sin(\omega t/R) \\ 0 & (1/\omega)\sin(\omega t/R) & \cos(\omega t/R) \end{bmatrix} = \begin{bmatrix} 0 & \cos(\omega t/R) & -\omega\sin(\omega t/R) \\ 0 & -(\omega/R)\sin(\omega t/R) & -(\omega^2/R)\cos(\omega t/R) \\ 0 & (1/R)\cos(\omega t/R) & -(\omega/R)\sin(\omega t/R) \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\omega^2/R \\ 0 & 1/R & 0 \end{bmatrix}\begin{bmatrix} 1 & (R/\omega)\sin(\omega t/R) & R\cos(\omega t/R) - R \\ 0 & \cos(\omega t/R) & -\omega\sin(\omega t/R) \\ 0 & (1/\omega)\sin(\omega t/R) & \cos(\omega t/R) \end{bmatrix} = \begin{bmatrix} 0 & \cos(\omega t/R) & -\omega\sin(\omega t/R) \\ 0 & -(\omega/R)\sin(\omega t/R) & -(\omega^2/R)\cos(\omega t/R) \\ 0 & (1/R)\cos(\omega t/R) & -(\omega/R)\sin(\omega t/R) \end{bmatrix}.$$

4. (a) The vector-matrix system $\dot{x} = Ax$ expands to $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_2$, giving

$$\ddot{x}_1 = \dot{x}_2 = -x_2 = -\dot{x}_1, \qquad \text{i.e.,} \qquad (*)\ \ddot{x}_1 + \dot{x}_1 = 0.$$

Equation $(*)$ is solved by $x_1 = e^{st}$ when $0 = s^2 + s$, i.e., when $s = 0$ or $s = -1$, giving

$$x_1(t) = \alpha + \beta e^{-t}, \qquad \alpha, \beta \in \mathbb{R}.$$

Recalling $x_2 = \dot{x}_1$ leads to the general solution of $\dot{x} = Ax$:

$$x(t) = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \qquad \alpha, \beta \in \mathbb{R}.$$

This has $x(0) = (\alpha + \beta, -\beta)$. Thus the initial condition $x(0) = \widehat{\mathbf{e}}_1$ leads to $\alpha = 1$, $\beta = 0$, and $x(t; \widehat{\mathbf{e}}_1) = (1, 0)$. The initial condition $x(0) = \widehat{\mathbf{e}}_2$ leads to $\alpha = 1$, $\beta = -1$, so $x(t; \widehat{\mathbf{e}}_2) = (1 - e^{-t}, e^{-t})$. In summary,

$$e^{At} = \left[x(t; \widehat{\mathbf{e}}_1) \middle| x(t; \widehat{\mathbf{e}}_2) \right] = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}.$$

(b) Expanding the standard formula

$$\begin{aligned} x(t) &= e^{At}\xi + \int_0^t e^{A(t-s)} Bu(s)\, ds \\ &= \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & 1 - e^{-(t-s)} \\ 0 & e^{-(t-s)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(s)\, ds \\ &= \begin{bmatrix} \xi_1 + \xi_2 - \xi_2 e^{-t} \\ \xi_2 e^{-t} \end{bmatrix} + \int_0^t \begin{bmatrix} 2 - e^{-(t-s)} \\ e^{-(t-s)} \end{bmatrix} u(s)\, ds \end{aligned}$$

produces the component equations

$$\begin{aligned} x_1(t) &= \xi_1 + \xi_2 - \xi_2 e^{-t} + \int_0^t \left(2 - e^{-(t-s)} \right) u(s)\, ds, \\ x_2(t) &= \xi_2 e^{-t} + \int_0^t e^{-(t-s)} u(s)\, ds. \end{aligned}$$

(c) When $(\xi_1, \xi_2) = (-1, 3)$, we get $\mathbf{x}(1) = \mathbf{0}$ exactly when u satisfies

$$(1) \qquad 0 = x_1(1) = 2 - 3e^{-1} + \int_0^1 \left(2 - e^{-(1-s)} \right) u(s)\, ds,$$

$$(2) \qquad 0 = x_2(1) = 3e^{-1} + \int_0^1 e^{-(1-s)} u(s)\, ds.$$

An equivalent system is produced by adding the equations, and by multiplying e into (2):

$$(1') \qquad 0 = 2 + 2\int_0^1 u(s)\, ds,$$

$$(2') \qquad 0 = 3 + \int_0^1 e^s u(s)\, ds.$$

To choose a function u that solves this system of two equations, one might try to introduce just two unknowns by choosing u from some two-parameter family of functions. Endless variety exists.

For example, one might try $u(t) = mt + b$ for some constants m, b. Integration by parts leads to

$$\begin{aligned} -1 &= \int_0^1 (ms + b)\, ds = \tfrac{1}{2}m + b, \quad \text{i.e.,} \quad m = -2(b+1), \\ -3 &= m \int_0^1 se^s\, ds + b \int_0^1 e^s\, ds = m \Big[(s-1)e^s \Big]_{s=0}^{1} + b(e-1) = m + b(e-1). \end{aligned}$$

Substituting the first equation into the second gives

$$-3 = -2b - 2 + (e-1)b, \quad \text{i.e.,} \quad -1 = b(e-3), \quad \text{i.e.,} \quad b = \frac{1}{3-e};$$

Hence $m = -2(b+1) = -2\left(\frac{4-e}{3-e}\right)$, and one possible control function is

$$u(t) = -2\left(\frac{4-e}{3-e}\right)t + \frac{1}{3-e}.$$

Alternatively, one could use a piecewise-constant assumption like

$$u(t) = \begin{cases} c_1, & \text{if } 0 \le t \le \frac{1}{2}, \\ c_2, & \text{if } \frac{1}{2} < t \le 1. \end{cases} \tag{$*$}$$

(Transition times other than $\frac{1}{2}$ can also work.) Easy integrals lead to the system

$$c_1 + c_2 = -4, \qquad (e^{1/2} - 1)c_1 + (e - e^{1/2})c_2 = -3,$$

so the constants that work with $(*)$ are

$$c_1 = \frac{3 + 4\sqrt{e} - 4e}{(\sqrt{e}-1)^2}, \quad c_2 = \frac{4\sqrt{e} - 7}{(\sqrt{e}-1)^2}.$$