Math 403 Problem Set 2

Due in class on Monday 24 September 2012

1. Evaluate e^{At} in each case:

$$\text{(i)}\quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}; \qquad \text{(ii)}\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}.$$

2. Find a 2×2 matrix-valued function $C(\cdot)$ with these properties:

$$\ddot{C}(t) + \begin{bmatrix} 26 & 10 \\ 10 & 26 \end{bmatrix} C(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ t \in \mathbb{R}; \qquad C(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \dot{C}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hint: The analogous scalar problem, $\ddot{c}(t) + \omega^2 c(t) = 0$ with $c(0) = 1$ and $\dot{c}(0) = 0$, has a well-known solution; guess and check an analogous matrix-valued solution.

3. Find a 2×2 matrix $M(k)$ such that $e^{M(k)} = \begin{bmatrix} 1+3k & k \\ k & 1+3k \end{bmatrix}$. Indicate the set of real numbers k for which the identity holds.

Remarks: Any logically correct approach is welcome. One possibility is to reason by analogy with the scalar case, where one could isolate M by taking the log of both sides in the desired identity. Recall that every complex number z obeying $|z| < 1$ satisfies

$$\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots.$$

If the problem's general case is daunting at first, start with $k = 1/8$.

4. (a) Prove that for matrix-valued functions A and B of appropriate dimensions,

$$\frac{d}{dt}\Big(A(t)B(t)\Big) = \frac{dA}{dt}B + A\frac{dB}{dt}.$$

(b) Given a smooth scalar-valued function α and a constant $n \times n$ matrix A_0, prove that

$$A(t) = \alpha(t)A_0 \implies \frac{d}{dt}\Big(A(t)^2\Big) = 2A(t)\dot{A}(t).$$

(c) Find a 2×2 matrix function $A(\cdot)$ for which $\frac{d}{dt}\Big(A(t)^2\Big) \neq 2A(t)\dot{A}(t)$.

(d) Consider this initial-value problem involving a time-varying $n \times n$ matrix A:

$$\dot{x}(t) = A(t)x(t), \qquad x(0) = \xi. \tag{1}$$

Prove:

(i) If $A(t) = \alpha(t)A_0$ as in (b) [in particular, if $n = 1$], then the unique solution of (1) is

$$x(t) = \exp\left(\int_0^t A(r)\,dr\right)\xi. \tag{2}$$

(ii) Formula (2) is not universal, even when $n = 2$. That is, there exists a 2×2-matrix-valued function $A(\cdot)$ for which formula (2) defines an arc x that *disobeys* (1). [*Hint*: The cleanest way to prove this is to display a specific function $A(\cdot)$.]

5. Consider the forced harmonic oscillator, for which the governing equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

We want to drive this system from the initial state $\xi = (1, 0)$ to the origin at time $t = 2\pi$.

(a) Is this transfer possible using some piecewise continuous function u?

(b) Is it possible using a piecewise *constant* control function of the form shown below?

$$u(t) = \begin{cases} u_1, & \text{if } 0 \le t < 2\pi/3, \\ u_2, & \text{if } 2\pi/3 \le t < 4\pi/3, \\ u_3, & \text{if } 4\pi/3 < t \le 2\pi. \end{cases}$$

If so, find all triples (u_1, u_2, u_3) that do the job.