M403(2012) Solutions—Problem Set 2

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1. (i) This matrix A has eigenvalues -1, $+1$ and $+4$. For eigenvectors,

$$\lambda = -1 \text{ gives } (1,0,-1); \quad \lambda = +1 \text{ gives } (1,-2,1); \quad \lambda = +4 \text{ gives } (1,1,1).$$

The eigenvector matrix

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \text{ has inverse } V^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 1 & -2 & 1 \\ 2 & 2 & 2 \end{bmatrix},$$

and thus the formula $e^{At} = Ve^{Dt}V^{-1}$, where $D = \operatorname{diag}(-1, 1, 4)$, gives

$$\begin{aligned} e^{At} &= \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}\begin{bmatrix} 3 & 0 & -3 \\ 1 & -2 & 1 \\ 2 & 2 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 3e^{-t}+e^{t}+2e^{4t} & -2e^{t}+2e^{4t} & -3e^{-t}+e^{t}+2e^{4t} \\ -2e^{t}+2e^{4t} & 4e^{t}+2e^{4t} & -2e^{t}+2e^{4t} \\ -3e^{-t}+e^{t}+2e^{4t} & -2e^{t}+2e^{4t} & 3e^{-t}+e^{t}+2e^{4t} \end{bmatrix}. \end{aligned}$$

(ii) This matrix A has eigenvalues 1, $1+2i$ and $1-2i$. For eigenvectors,

$$\lambda = 1 \text{ gives } (2,-3,2); \quad \lambda = 1+2i \text{ gives } (0,i,1); \quad \lambda = 1-2i \text{ gives } (0,-i,1).$$

The eigenvector matrix

$$V = \begin{bmatrix} 2 & 0 & 0 \\ -3 & i & -i \\ 2 & 1 & 1 \end{bmatrix} \text{ has inverse } V^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 \\ -2-3i & -2i & 2 \\ -2+3i & 2i & 2 \end{bmatrix},$$

and thus the formula $e^{At} = Ve^{Dt}V^{-1}$, where $D = \operatorname{diag}(1, 1+2i, 1-2i)$, gives

$$\begin{aligned} e^{At} &= \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 \\ -3 & i & -i \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{(1+2i)t} & 0 \\ 0 & 0 & e^{(1-2i)t} \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ -2-3i & -2i & 2 \\ -2+3i & 2i & 2 \end{bmatrix} \\ &= \frac{e^{t}}{2}\begin{bmatrix} 2 & 0 & 0 \\ -3+3\cos 2t+2\sin 2t & 2\cos 2t & -2\sin 2t \\ 2-2\cos 2t+3\sin 2t & 2\sin 2t & 2\cos 2t \end{bmatrix}. \end{aligned}$$

2. The matrix $P = \begin{bmatrix} 26 & 10 \\ 10 & 26 \end{bmatrix}$ is symmetric, so it is orthogonally diagonalizable. Its row sums are equal (both come to 36), so one eigenvector is $v_1 = (1,1)$, with corresponding eigenvalue $\lambda_1 = 36$. The trace of P is the sum of its eigenvalues, so $\lambda_2 = \operatorname{tr}(P) - 36 = 16$. It is not too much trouble to find an eigenvector for λ_2: my choice is $v_2 = (-1,1)$. This calculation leads to the eigenvector matrix

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{ whose inverse is } V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

There are two ways to complete the problem. The first is to follow the clue given in the statement. Using V, we can find a matrix square root of P, namely, $W = V\sqrt{D}V^{-1}$, where $D = \operatorname{diag}(36, 16)$:

$$W = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}. \qquad (*)$$

(Indeed, this solution of $W^2 = P$ is almost guessable.) Analogy with the scalar equation suggests that $C(t) = \cos(Wt)$ should be the right answer; it remains only to interpret and calculate the matrix cosine. For this, it suffices to realize that a diagonalization of W already appears in equation $(*)$, so that the power series formula for the cosine function (valid for all real inputs) gives

$$\begin{aligned} C(t) = \cos(Wt) &= V \operatorname{diag}\left(\cos(6t), \cos(4t)\right) V^{-1} \\ &= \frac{1}{2}\begin{bmatrix} \cos(6t)+\cos(4t) & \cos(6t)-\cos(4t) \\ \cos(6t)-\cos(4t) & \cos(6t)+\cos(4t) \end{bmatrix}. \end{aligned} \qquad (**)$$

A second approach is to use the change of variables implicit in the eigenvector matrix V to introduce a new unknown $X(t) = V^{-1}C(t)V$, in terms of which the original initial-value problem becomes diagonal: $C = VXV^{-1}$, so

$$\begin{aligned} V\ddot{X}(t)V^{-1} + \left[VDV^{-1}\right]\left[VX(t)V^{-1}\right] = 0, \quad VX(0)V^{-1} = I,\ V\dot{X}(0)V^{-1} = 0, \\ \iff \quad \ddot{X}(t) + DX(t) = 0, \quad X(0) = I,\ \dot{X}(0) = 0. \end{aligned}$$

The latter equation is completely decoupled. Solving it componentwise leads to the diagonal matrix $X(t) = \cos(Dt) = \operatorname{diag}\left(\cos(6t), \cos(4t)\right)$, from which the desired solution can be recovered by reversing the change of variables: $C(t) = VX(t)V^{-1}$. The result is, of course, the same as $(**)$.

3. Consider the matrix $A(k)$ defined by $e^{M(k)} = I + A(k)$, i.e.,

$$A(k) = \begin{bmatrix} 3k & k \\ k & 3k \end{bmatrix}.$$

Two linearly independent eigenvectors of $A(k)$ are $\mathbf{u} = (1,1)$, with eigenvalue $4k$, and $\mathbf{w} = (1,-1)$, with eigenvalue $2k$. We use these to define the eigenvector matrix

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{ where } V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

This diagonalizes not only $A(k)$, but also $I + A(k)$, since

$$I + A(k) = I + VD(k)V^{-1} = V(I + D(k))V^{-1} \iff D(k) = V^{-1}A(k)V = \begin{bmatrix} 4k & 0 \\ 0 & 2k \end{bmatrix}.$$

So the matrix equation we hope to solve amounts to

$$e^{M(k)} = V(I + D(k))V^{-1}, \qquad \text{i.e.,} \qquad V^{-1}e^{M(k)}V = I + D(k) = \begin{bmatrix} 1+4k & 0 \\ 0 & 1+2k \end{bmatrix}.$$

Here the similarity transformation applies on the left, and the simple exponential formula for diagonal matrices lets us rewrite the right. The desired identity becomes

$$\exp\left(V^{-1}M(k)V\right) = \exp\left(\begin{bmatrix} \log(1+4k) & 0 \\ 0 & \log(1+2k) \end{bmatrix}\right).$$

The simplification on the right make sense whenever $k > -1/4$, and for these values the solution is

$$M(k) = V\begin{bmatrix} \log(1+4k) & 0 \\ 0 & \log(1+2k) \end{bmatrix}V^{-1} = \frac{1}{2}\begin{bmatrix} \log(1+4k)+\log(1+2k) & \log(1+4k)-\log(1+2k) \\ \log(1+4k)-\log(1+2k) & \log(1+4k)+\log(1+2k) \end{bmatrix}.$$

Remarks: The same diagonalization scheme can produce the same answer starting from

$$\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots.$$

4. (a) Suppose $A(t)$ is $m \times n$ and $B(t)$ is $n \times p$. Then just use the definition of the matrix product and the product rule for differentiating scalar functions:

$$\begin{aligned}\frac{d}{dt}\left[A(t)B(t)\right]_{ij} &= \frac{d}{dt}\sum_{k=1}^{n} a_{ik}(t)b_{kj}(t) \\ &= \sum_{k=1}^{n}\left[\dot a_{ik}(t)b_{kj}(t) + a_{ik}(t)\dot b_{kj}(t)\right] \\ &= \left[\dot A(t)B(t) + A(t)\dot B(t)\right]_{ij}\end{aligned}$$

(b) Taking $A = B = M$ in (a) gives

$$\frac{d}{dt}\left(M(t)^2\right) = \dot M(t)M(t) + M(t)\dot M(t),$$

so

$$\frac{d}{dt}\left(M(t)^2\right) - 2M(t)\dot M(t) = \dot M(t)M(t) - M(t)\dot M(t). \qquad (\dagger)$$

Thus the difference between the left-hand side and the zero matrix is precisely the amount by which the matrix M fails to commute with its time derivative $\dot M$. For the special case where $M(t) = \alpha(t)A_0$, the matrix $\dot M = \dot\alpha A_0$ clearly commutes with M, so the right side in (†) is zero.

(c) For a 2×2 matrix $M(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$, line (†) implies

$$\frac{d}{dt}\left(M(t)^2\right) - 2M(t)\dot M(t) = \begin{bmatrix} \dot b c - b\dot c & (\dot a b - a\dot b) + (\dot b d - b\dot d) \\ (a\dot c - \dot a c) + (c\dot d - \dot c d) & b\dot c - \dot b c \end{bmatrix}.$$

The right hand side is nonzero for many choices of a, b, c, and d—including, for example, $a(t) = 0 = d(t)$ and $b(t) = t$, $c(t) = t^2$. (Simple choices are best, but part (b) shows that they can't be too simple—that is, a time-varying multiple of a constant matrix cannot provide the desired example.)

(d) Since the key point in (c) is to find a matrix that fails to commute with its derivative, and since this is likely to be the same sort of behaviour that makes trouble for the case where $M(t) = \int_0^t A(r)\,dr$, we take $M(t)$ from part (b) and consider $A(t) = \dot M(t) = \begin{bmatrix} 0 & 1 \\ 2t & 0 \end{bmatrix}$. We then have

$$\exp\left(\int_0^t A(r)\,dr\right) = e^{M(t)}.$$

To calculate the latter, compute instead $e^{M(k^2)t}$ for $k > 0$, $t = 1$. This is a matrix whose columns are particular solutions of the ODE system

$$\left.\begin{aligned}\dot x_1 &= k^2 x_2, \\ \dot x_2 &= k^4 x_1,\end{aligned}\right\} \iff \left\{\begin{aligned}\ddot x_1 - k^6 x_1 &= 0, \\ x_2 &= \dot x_1/k^2.\end{aligned}\right.$$

The general solution here is $x_1(t) = Ae^{k^3 t} + Be^{-k^3 t}$, $x_2(t) = Ake^{k^3 t} - Bke^{-k^3 t}$. Setting the initial condition $(x_1(0), x_2(0)) = (1, 0)$ fixes $A = \frac{1}{2} = B$, and thus gives the first column of the desired matrix $e^{M(k^2)} = e^{M(k^2)t}\Big|_{t=1}$:

$$e^{M(k^2)} = \begin{bmatrix} \frac{1}{2}e^{k^3} + \frac{1}{2}e^{-k^3} & \frac{1}{2k}e^{k^3 t} - \frac{1}{2k}e^{-k^3 t} \\ \frac{1}{2}ke^{k^3} - \frac{1}{2}ke^{-k^3} & \frac{1}{2}e^{-k^3 t} + \frac{1}{2}e^{k^3 t} \end{bmatrix}.$$

Now putting $k^2 = t$ for $t > 0$ and taking the initial vector $\xi = (1, 0)$, we find that the erroneous formula $x(t) = e^{\int_0^t A(r)\,dr}\xi$ suggests that

$$x(t) = \tfrac{1}{2}\exp(t^{3/2}) + \tfrac{1}{2}\exp(-t^{3/2})$$

should solve the second-order linear ODE $t\ddot{x}_1 - \dot{x}_1 - t^4 x_1 = 0$. It is simple to check that this is not the case, so this matrix M provides a suitable counterexample.

5. (a) Yes: Given any positive length of time, some piecewise continuous control function will steer the oscillator from any initial position to the origin in that time. This is true because the system is controllable, which is clear because it is in controllable canonical form.

(b) We proved in class that for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ one has } e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

So for any piecewise continuous control function u, the evolution starting at $\xi = (1, 0)$ is given by

$$\begin{aligned} x(t) &= e^{At}\xi + \int_0^t e^{A(t-s)} Bu(s)\,ds \\ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{bmatrix} \begin{bmatrix} 0 \\ u(s) \end{bmatrix} ds \\ &= \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + \int_0^t \begin{bmatrix} u(s)\sin(t-s) \\ u(s)\cos(t-s) \end{bmatrix} ds. \end{aligned}$$

In particular,

$$\begin{bmatrix} x_1(2\pi) \\ x_2(2\pi) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^{2\pi} \begin{bmatrix} -u(s)\sin s \\ u(s)\cos s \end{bmatrix} ds.$$

To treat the piecewise constant control suggested in the problem, we note that

$$\int \begin{bmatrix} -\sin s \\ \cos s \end{bmatrix} ds = \begin{bmatrix} \cos s \\ \sin s \end{bmatrix} + \text{const.},$$

so

$$\begin{aligned} \begin{bmatrix} x_1(2\pi) \\ x_2(2\pi) \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} \cos s \\ \sin s \end{bmatrix}\Bigg|_0^{2\pi/3} + u_2 \begin{bmatrix} \cos s \\ \sin s \end{bmatrix}\Bigg|_{2\pi/3}^{4\pi/3} + u_3 \begin{bmatrix} \cos s \\ \sin s \end{bmatrix}\Bigg|_{4\pi/3}^{2\pi} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} -3/2 \\ \sqrt{3}/2 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix} + u_3 \begin{bmatrix} 3/2 \\ \sqrt{3}/2 \end{bmatrix}. \end{aligned}$$

At this point it is obvious that the right hand side can be made to equal $(0, 0)$ by some choice of u_1, u_2, and u_3, because these are simply the real coefficients of three vectors which span $\mathbb{R}^2$. A complete list of vectors that do this job is easy to get by solving two linear equations in three unknowns: the result is $u_1 = 2/3 + z$, $u_2 = 1/3 + z$, $u_3 = z$, for any real z.