# Math 403 Problem Set 3

Due in class on Monday 01 October 2012

**1.** The famous "broom-balancing problem" involves a cart of mass $M$ running along a straight horizontal track under the influence of a force $u$. Friction proportional to the cart's velocity opposes its motion; the friction coefficient is $k$. On top of the cart is a frictionless hinge connected to a lightweight rod of length $\ell$, with a concentrated mass $m$ at its free end. Writing $\delta$ for the cart's horizontal displacement from the origin and $\phi$ for the angle between the rod and the vertical leads to this system of equations:

$$\begin{aligned}&(M+m)\ddot{\delta} + m\ell\ddot{\phi}\cos\phi - m\ell\dot{\phi}^2\sin\phi + k\dot{\delta} = u,\\ &\ell\ddot{\phi} - g\sin\phi + \ddot{\delta}\cos\phi = 0.\end{aligned}$$

(Here $g > 0$ is a constant representing the acceleration due to gravity.) In the following steps, assume $M, m, k, \ell, g = 1$ for simplicity.

(a) Let $\mathbf{x} = (\delta, \dot{\delta}, \phi, \dot{\phi})$. Find a (nonlinear) system of controlled differential equations for $\mathbf{x}$.

(b) Show that the vector $\mathbf{x} = \mathbf{0}$, which represents the cart parked at the origin with the rod perfectly vertical and stationary, is an equilibrium state.

(c) Linearize the system in (a) about the equilibrium point in (b). The desired outcome is a *linear* system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \tag{$*$}$$

closely related to the one in part (a). Use the same variable names ($\mathbf{x}, \delta, \phi$, etc.) as in (a), even though the new system is only an approximation of the original one.

(d) Find the eigenvalues of the linearized system matrix $A$ and comment on the stability of $(*)$ when $u \equiv 0$.

**2.** Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} u(t).$$

(a) Is the system controllable?

(b) Compute the zero-input response for $x(0) = \xi = (\xi_1, \xi_2, \xi_3)$.

(c) Give a complete description of the attainable set $\mathcal{A}(1; (1, 0, 1))$.

**3.** Consider the controlled differential equation $\dot{x} = Ax + Bu$, in which $u$ is a scalar input, so that $B$ is a column vector in $\mathbb{R}^n$. Assume that the $n \times n$ matrix $A$ has distinct eigenvalues, and let $P$ be a matrix whose columns form a set of linearly independent eigenvectors for $A$. Prove "Gilbert's controllability criterion": The matrix pair $(A, B)$ is controllable if and only if the column vector $\widetilde{B} = P^{-1}B$ has no zero entries.

**4.** Find the set of ordered pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which this linear system is controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & \alpha \\ 1 & 2 & \beta \\ \alpha & \beta & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

**5.** Consider the control system in **Question 4** with $\alpha = 1$ and $\beta = 0$. Use any method to design a feedback law $u = F\mathbf{x}$ for which the closed-loop system eigenvalues are $0, -1, -2$.

**6.** Consider this control system with a scalar input $u$:

$$(*) \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} u.$$

(i) Find all values of $a$ and $b$ for which system $(*)$ is controllable.

(ii) Assuming $b \neq 0$, find all vectors $(v_1, v_2)$ such that using the feedback law $u = \mathbf{v} \bullet \mathbf{x}$ in $(*)$ produces a system in which every trajectory obeys $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. Give a general solution in terms of $a$ and $b$, then sketch the appropriate region of the $(v_1, v_2)$-plane for the special cases $a = 2$, $b = 1$.