M403(2012) Solutions—Problem Set 3

1. (a) With $\mathbf{x} = (\delta, \dot{\delta}, \phi, \dot{\phi})$, we have $\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$, and

$$(1) \qquad (M+m)\dot{x}_2 + m\ell\dot{x}_4 \cos x_3 - m\ell x_4^2 \sin x_3 + kx_2 = u,$$

$$(2) \qquad \ell\dot{x}_4 - g\sin x_3 + \dot{x}_2 \cos x_3 = 0.$$

Solving (2) for $\ell\dot{x}_4 = g\sin x_3 - \dot{x}_2\cos x_3$ and substituting in (1) leads to an equation linear in $\dot{x}_2$:

$$(M+m)\dot{x}_2 = -kx_2 - m\left[g\sin x_3 - \dot{x}_2\cos x_3\right]\cos x_3 + m\ell x_4^2 \sin x_3 + u$$

$$(M+m-m\cos^2 x_3)\dot{x}_2 = -kx_2 + (m\ell x_4^2 - mg\cos x_3)\sin x_3 + u.$$

Using $\sin^2\phi + \cos^2\phi = 1$ and back-substituting into (2) leads to

$$\dot{x}_2 = \frac{(m\ell x_4^2 - mg\cos x_3)\sin x_3 - kx_2 + u}{M + m\sin^2 x_3},$$

$$\begin{aligned}\dot{x}_4 &= \frac{g}{\ell}\sin x_3 - \frac{\cos x_3}{\ell}\left[\frac{(m\ell x_4^2 - mg\cos x_3)\sin x_3 - kx_2 + u}{M + m\sin^2 x_3}\right] \\ &= \frac{Mg\sin x_3 + mg\sin^3 x_3 - \cos x_3(m\ell x_4^2 - mg\cos x_3)\sin x_3 - kx_2 + u}{\ell(M + m\sin^2 x_3)} \\ &= \frac{Mg\sin x_3 + mg\sin x_3 - m\ell x_4^2\cos x_3\sin x_3 + kx_2\cos x_3 - u\cos x_3}{\ell(M + m\sin^2 x_3)}\end{aligned}$$

These equations allow us to express the given dynamics as a controlled fourth-order system. In the special case where all physical constants have magnitude 1, the system reduces to

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{(x_4^2 - \cos x_3)\sin x_3 - x_2 + u}{1 + \sin^2 x_3}, \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{2\sin x_3 - x_4^2\cos x_3\sin x_3 + x_2\cos x_3 - u\cos x_3}{1 + \sin^2 x_3}\end{aligned}$$

(b) To show that $\mathbf{x} = \mathbf{0}$ is an equilibrium point when $u = 0$, it suffices to note that plugging $\mathbf{x} = \mathbf{0}$ into the right side of the differential equation above produces the zero vector. We can do a little more, by doing the work in two stages. Let $f(\mathbf{x}, u)$ denote the general function defined by the RHS, and plug in just $x_3 = 0$ and $u = 0$ at first. Then

$$f((x_1, x_2, 0, x_4), 0) = \left(x_2,\ -\frac{kx_2}{M},\ x_4,\ \frac{kx_2}{\ell M}\right).$$

The vector on the right will be $\mathbf{0}$ whenever $x_2 = 0$ and $x_4 = 0$. This confirms the desired statement, but has another interesting feature: there are no restrictions on x_1. So in fact there is a whole family of equilibrium solutions for $u = 0$, with the form $(c, 0, 0, 0)$ for a real constant c. This makes good sense from a physical viewpoint: everything about the system model is insensitive to the choice of origin along the horizontal axis, so it should be kinematically equivalent to park the car at any location with the broom in the vertical position.

(c) The two interesting component functions in the dynamics are

$$f_2(\mathbf{x}, u) = \frac{(m\ell x_4^2 - mg\cos x_3)\sin x_3 - kx_2 + u}{M + m\sin^2 x_3},$$

$$f_4(\mathbf{x}, u) = \frac{Mg\sin x_3 + mg\sin x_3 - m\ell x_4^2\cos x_3\sin x_3 + kx_2\cos x_3 - u\cos x_3}{\ell(M + m\sin^2 x_3)}.$$

Calculation gives the general formulas and evaluation results below:

$$D_{\mathbf{x}} f_2(\mathbf{x},u) = \begin{bmatrix} 0 & -\dfrac{k}{M+m\sin^2 x_3} & \left(\dfrac{\partial f_2}{\partial x_3}\right) & \dfrac{2\ell m x_4}{M+m\sin^2 x_3} \end{bmatrix}, \qquad D_u f_2(\mathbf{x},u) = \frac{1}{M+m\sin^2 x_3},$$

$$D_{\mathbf{x}} f_2(\mathbf{0},0) = \begin{bmatrix} 0 & -\dfrac{k}{M} & -\dfrac{mg}{M} & 0 \end{bmatrix}, \qquad D_u f_2(\mathbf{0},0) = \frac{1}{M},$$

$$D_{\mathbf{x}} f_4(\mathbf{x},u) = \begin{bmatrix} 0 & \dfrac{k\cos x_3}{\ell(M+m\sin^2 x_3)} & \left(\dfrac{\partial f_4}{\partial x_3}\right) & \dfrac{-2\ell m x_4\cos x_3}{\ell(M+m\sin^2 x_3)} \end{bmatrix}, \qquad D_u f_4(\mathbf{x},u) = -\frac{\cos x_3}{\ell(M+m\sin^2 x_3)},$$

$$D_{\mathbf{x}} f_4(\mathbf{0},0) = \begin{bmatrix} 0 & \dfrac{k}{\ell M} & \dfrac{(M+m)g}{\ell} & 0 \end{bmatrix}, \qquad D_u f_4(\mathbf{0},0) = -\frac{1}{\ell M}.$$

Here is a shortcut for calculating the two partial derivatives $(\partial f_k/\partial x_3)$ not shown explicitly above. Both involve functions of the form $h(x_3)/(a+b\sin^2(x_3))$ for some constants a and b, and the quotient rule in abstract notation gives

$$\left[\frac{\partial}{\partial x_3}\frac{h(x_3)}{(a+b\sin^2(x_3))}\right]_{x_3=0} = \left[\frac{h'(x_3)(a+b\sin^2(x_3)) - h(x_3)[2b\sin(x_3)\cos(x_3)]}{(a+b\sin^2(x_3))^2}\right]_{x_3=0} = \frac{h'(0)}{a}.$$

Making appropriate choices for a, b, and h in this framework produces the results announced above. These provide the two interesting equations in the linearization we seek; the equations involving component-functions f_1 and f_3 are linear already, so we don't show their calculations. The linearized dynamics are $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, where $A = D_{\mathbf{x}} f(\mathbf{0},0)$ and $B = D_u f(\mathbf{0},0)$ are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -k/M & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & k/(M\ell) & (M+m)g/(M\ell) & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(M\ell) \end{bmatrix}.$$

When all physical parameters have magnitude 1, these matrices become

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{$*$}$$

(d) The characteristic polynomial of the system matrix A is

$$\det(sI - A) = \begin{vmatrix} s & -1 & 0 & 0 \\ 0 & s+1 & 1 & 0 \\ 0 & 0 & s & -1 \\ 0 & -1 & -2 & s \end{vmatrix} = s^4 + s^3 - 2s^2 - s = s(s^3 + s^2 - 2s - 1).$$

Therefore one eigenvalue is $\lambda = 0$, and its corresponding eigenvector is $\widehat{\mathbf{e}}_1$. This corresponds to the neutral stability of shifting the equilibrium point sideways along the x-axis. The other 3 eigenvalues are the zeros of $g(s) \overset{\text{def}}{=} s^3 + s^2 - 2s - 1$. Locating these precisely is difficult, but we can say something meaningful quite easily. Since $g(0) = -1 < 0$ and $g(s) \to +\infty$ as $s \to +\infty$, the function g must have at least one positive real root. Therefore the matrix A must have at least one positive real eigenvalue. That means that the input $u = 0$ gives us a nonlinear dynamical system for which the equilibrium point $\mathbf{x} = \mathbf{0}$ is *unstable*.

There are various ways to approximate the eigenvalues of A. One could search for the zeros of g using a graphing calculator, or enter the matrix A into Matlab and give the command `eig(A)`. Any method is acceptable; the computed eigenvalues are

$$\sigma(A) = \{-1.8019, -0.4450, 0, 1.2470\}.$$

Remark. The simple small-angle approximation $\sin x_3 \approx x_3$, followed by taking $x_i x_j \approx 0$, etc., provides a rather ad-hoc but undeniably effective alternative route to the linearized equations derived "the hard way" in part (c):

$$\dot{x}_2 \approx \frac{-mgx_3 - kx_2 + u}{M},$$
$$\dot{x}_4 \approx \left(\frac{k}{M\ell}\right)x_2 + \frac{(M+m)g}{\ell}x_3 - \frac{u}{M\ell}.$$

Combining approximations like this can ultimately be justified by the kind of trick presented in part (c) above: one applies some differentiation rule in semi-abstract notation to show that evaluation at the point of interest will put a zero in the spot the informal argument was set to ignore anyway.

2. (a) The controllability matrix for this system is

$$\mathcal{C} = \begin{bmatrix} 3 & 5 & 11 \\ -2 & -6 & -18 \\ -1 & 1 & 7 \end{bmatrix}.$$

Since $[1\ 1\ 1]\mathcal{C} = [0\ 0\ 0]$, $\mathcal{C}$ does not have full rank. This system is not controllable.

(b) The zero-input response corresponding to initial state ξ is $e^{At}\xi$. Here we have

$$|\lambda I - A| = (\lambda - 1)\left[(\lambda-2)(\lambda-1) - 2\right] = \lambda(\lambda-1)(\lambda-3),$$

so the eigenvalues of A are 0, 1, and 3.

$$\text{For } \lambda_1 = 0, \text{ an eigenvector is } (1,1,1);$$
$$\text{for } \lambda_2 = 1, \text{ an eigenvector is } (1,0,-1);$$
$$\text{for } \lambda_3 = 3, \text{ an eigenvector is } (1,-2,1).$$

Therefore $P^{-1}AP = \mathrm{diag}(0,1,3)$ for the choice

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \quad \Leftrightarrow \quad P^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}.$$

It follows that $e^{At} = P\,\mathrm{diag}(1, e^t, e^{3t})P^{-1}$, i.e.,

$$e^{At} = \frac{1}{6}\begin{bmatrix} 2+3e^t+e^{3t} & 2-2e^{3t} & 2-3e^t+e^{3t} \\ 2-2e^{3t} & 2+4e^{3t} & 2-2e^{3t} \\ 2-3e^t+e^{3t} & 2-2e^{3t} & 2+3e^t+e^{3t} \end{bmatrix};$$

the zero-input response is $x(t;\xi) = e^{At}\xi$.

(c) We have shown that $\mathcal{A}(t;\xi) = e^{At}\xi + \mathcal{A}(t,0)$ for any $\xi \in \mathbb{R}^n$ and $t \geq 0$. Moreover, for $t > 0$, $\mathcal{A}(t,0) = \mathrm{Image}(\mathcal{C})$ is the subspace of $\mathbb{R}^n$ spanned by the columns of the controllability matrix. Column reduction of $\mathcal{C}$ produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix},$$

whose columns span the space $\{(r, s, -s-r) \,:\, r, s \in \mathbb{R}\}$. Therefore

$$\mathcal{A}(1;(1,0,1)) = \left\{ \frac{1}{3}\begin{pmatrix} 2+e^3 \\ 2-2e^3 \\ 2+e^3 \end{pmatrix} + \begin{pmatrix} r \\ s \\ -s-r \end{pmatrix} \,:\, r,s \in \mathbb{R} \right\}.$$

Geometrically, this is the plane $x + y + z = 2$.

3. We know that the matrix pair (A, B) is controllable if and only if the related pair $(\widetilde{A}, \widetilde{B})$ is controllable, where

$$\widetilde{A} = P^{-1}AP, \qquad \widetilde{B} = P^{-1}B.$$

The point here is that $\widetilde{A}$ is a diagonal matrix, whose entries are the n distinct eigenvalues of A. Thus, for each $k = 0, \ldots, n-1$,

$$(\widetilde{A})^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^k & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}.$$

It follows that the controllability matrix for $(\widetilde{A}, \widetilde{B})$ is

$$\begin{aligned}\widetilde{C} &= \begin{bmatrix} \widetilde{b}_1 & \lambda_1\widetilde{b}_1 & \lambda_1^2\widetilde{b}_1 & \cdots & \lambda_1^{n-1}\widetilde{b}_1 \\ \widetilde{b}_2 & \lambda_2\widetilde{b}_2 & \lambda_2^2\widetilde{b}_2 & \cdots & \lambda_2^{n-1}\widetilde{b}_2 \\ \vdots & & & & \vdots \\ \widetilde{b}_n & \lambda_n\widetilde{b}_n & \lambda_n^2\widetilde{b}_n & \cdots & \lambda_n^{n-1}\widetilde{b}_n \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{b}_1 & 0 & 0 & \cdots & 0 \\ 0 & \widetilde{b}_2 & 0 & \cdots & 0 \\ 0 & 0 & \widetilde{b}_3 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \widetilde{b}_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & \lambda_3^{n-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}.\end{aligned}$$

Hence $\det(\widetilde{C}) = \det(\mathrm{diag}(\widetilde{B}))\det(V)$, where V denotes the matrix on the far right. This is a so-called "Vandermonde matrix", which is known to be invertible if and only if the numbers in its second column are distinct. Since we have assumed this, we have $\det(\widetilde{C}) \neq 0$ if and only if $\det(\mathrm{diag}(\widetilde{B})) \neq 0$, i.e., if and only if each entry of the column vector $\widetilde{B}$ is nonzero.

A nice way to see that V is invertible is to imagine the meaning of the equation $V\mathbf{a} = 0$, where $\mathbf{a} \in \mathbb{R}^n$ has subscripts starting from 0:

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}. \qquad (*)$$

The n component-equations in $(*)$ require $0 = p(\lambda_k)$ for each $k = 1, 2, \ldots, n$, where

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1}z^{n-1}.$$

But the only way a polynomial of degree $n-1$ or less can have n distinct zeros is if all its coefficients are zero. Hence $(*)$ forces $\mathbf{a} = \mathbf{0}$, so matrix V is invertible.

4. Build the Kalman controllability matrix

$$\mathcal{C} = [B \,|\, AB \,|\, A(AB)] = \begin{bmatrix} 0 & \alpha & 2\alpha + \beta - 2\alpha \\ 0 & \beta & \alpha + 2\beta - 2\beta \\ 1 & -2 & \alpha^2 + \beta^2 + 4 \end{bmatrix} = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & \beta & \alpha \\ 1 & -2 & \alpha^2 + \beta^2 + 4 \end{bmatrix}.$$

Work out $\det(\mathcal{C})$ by expanding along the first column:

$$\det(\mathcal{C}) = \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} = \alpha^2 - \beta^2.$$

The given system is controllable if and only if $\alpha^2 \neq \beta^2$.

5. If $F = [r\ s\ t]$, then we are dealing with

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -2 \end{bmatrix}, \qquad BF = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} r & s & t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r & s & t \end{bmatrix}.$$

To find the characteristic polynomial of $A + BF$, expand down column 3:

$$\begin{aligned} p(\lambda) = \det(A + BF - \lambda I) &= \det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1+r & s & t-2-\lambda \end{pmatrix} \\ &= \begin{vmatrix} 1 & 2-\lambda \\ 1+r & s \end{vmatrix} + (t-2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= s + (\lambda-2)(r+1) + (t-2-\lambda)(\lambda^2 - 4\lambda + 3) \\ &= -\lambda^3 + (t+2)\lambda^2 + (6 - 4t + r)\lambda + (3t + s - 2r - 8). \end{aligned}$$

The characteristic polynomial we want has zeros at $0, -1, -2$, so it is

$$-\lambda(\lambda+1)(\lambda+2) = -\lambda^3 - 3\lambda^2 - 2\lambda.$$

Matching coefficients above produces a system of equations for r, s, t:

$$t + 2 = -3, \qquad 6 - 4t + r = -2, \qquad 3t + s - 2r - 8 = 0.$$

Solving from left-to-right gives $t = -5$, $r = 4t - 8 = -28$, $s = 8 + 2r - 3t = -33$. In summary,

$$F = \begin{bmatrix} -28 & -33 & -5 \end{bmatrix}.$$

Alternatively, follow the recipe given in class. Set $[r, s, t] = [0, 0, 0]$ above to get

$$\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0, \quad \text{with} \quad [a_0, a_1, a_2] = [-2, -6, 8].$$

Push these into a matrix V and use the controllability matrix $\mathcal{C}$ to find

$$P = \mathcal{C}V = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} -6 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 4 & 5 & 1 \end{bmatrix}.$$

Now the system with matrices $\widetilde{A} = P^{-1}AP$, $\widetilde{B} = P^{-1}B$ is in controllable canonical form and applying feedback $\widetilde{F} = [\widetilde{r}\ \widetilde{s}\ \widetilde{t}]$ will make its characteristic polynomial

$$\det(sI - (\widetilde{A} + \widetilde{B}\widetilde{F})) = s^3 + (-2 - \widetilde{t})s^2 + (-6 - \widetilde{s})s + (8 - \widetilde{r}).$$

Matching the desired polynomial, $s(s+1)(s+2)$, requires

$$-2 - \widetilde{t} = 3, \ -6 - \widetilde{s} = 2, \ 8 - \widetilde{r} = 0, \qquad \text{i.e.,} \qquad \widetilde{t} = -5, \ \widetilde{s} = -8, \ \widetilde{r} = 8.$$

Hence $\widetilde{F} = [8\ \ -8\ \ -5]$, and the feedback matrix we want is

$$F = \widetilde{F}P^{-1} = \begin{bmatrix} -28 & -33 & -5 \end{bmatrix}.$$

6. (i) The controllability matrix is

$$\mathcal{C} = [B \,|\, AB] = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}.$$

Its determinant is $-b^2$, so it has full rank if and only if $b \neq 0$. Thus the system is controllable iff $b \neq 0$; there are no restrictions on a.

(ii) The closed-loop system described in the question has dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} av_1 & av_2 \\ bv_1 & bv_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} av_1 & 1+av_2 \\ bv_1 & bv_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

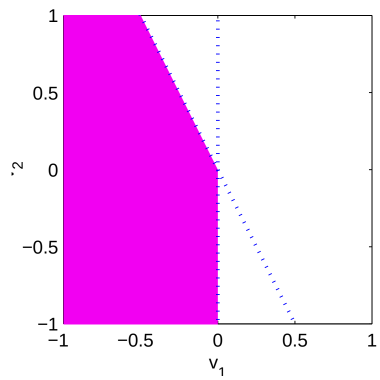
The eigenvalues of the coefficient matrix here are given by

$$0 = (av_1 - \lambda)(bv_2 - \lambda) - (bv_1)(1 + av_2) = \lambda^2 - [av_1 + bv_2]\,\lambda - bv_1.$$

We want both to have negative real part. This happens if and only if both

$$av_1 + bv_2 < 0 \qquad \text{and} \qquad bv_1 < 0.$$

Each of these inequalities describes an open half-space having the origin as a boundary point. Their intersection is the desired region of the (v_1, v_2)-plane. A sketch for the case where $a = 2$, $b = 1$ appears below.



Aside: For the general polynomial $p(s) = s^2 + bs + c$, the roots obey $2s = -b \pm \sqrt{b^2 - 4c}$. In cases where $b^2 - 4c \leq 0$, the roots have negative real part iff $b > 0$. In cases where $b^2 - 4c > 0$, the larger root is $-b + \sqrt{b^2 - 4c}$. It will be negative if and only if $\sqrt{b^2 - 4c} < b$, i.e., if and only if $b > 0$ and $c > 0$. A nice way to combine the cases is to make a Cartesian plane with axes b and c. Above the parabola $c = b^2/4$, p has complex roots and $b > 0$ is required. Below that parabola, p has real roots and both $b > 0$ and $c > 0$ are essential. Thus both roots of p will have negative real parts if and only if both $b > 0$ and $c > 0$.