Math 403 Problem Set 4

Due in class on Monday 15 October 2012

1. Given a real matrix $A \in \mathbb{R}^{n\times n}$, let $t \mapsto \mathbf{x}(t;\xi)$ denote the unique solution of

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t),\ t > 0; \qquad \mathbf{x}(0) = \xi.$$

Sometimes it is impossible to measure all n components of the state vector directly, and one has access only to the p components of an "observation" $\mathbf{y} \stackrel{\text{def}}{=} C\mathbf{x}$, where C is a given $p \times n$ matrix. Let $\mathbf{y}(t;\xi) = C\mathbf{x}(t;\xi)$.

Prove that the following are equivalent:

(a) Identical observation histories guarantee identical initial states, i.e.,

$$\left[\text{ for some } T > 0, \text{ one has } \mathbf{y}(t;\xi) = \mathbf{y}(t;\eta) \quad \forall t \in (0,T)\right] \quad \Longrightarrow \quad \xi = \eta.$$

(b) Different initial states will cause differences in observation values after arbitrarily short times, i.e.,

$$\xi \neq \eta \quad \Longrightarrow \quad \left[\text{ for each } T > 0, \text{ there is some } t \in (0,T) \text{ such that } \mathbf{y}(t;\xi) \neq \mathbf{y}(t;\eta)\right].$$

(c) The matrix pair (A^T, C^T) is controllable.

Statements (a) and (b) explain why the matrix pair (A, C) is called *observable* in situations (a)–(c).

Hint: Consider the subspace of $\mathbb{R}^n$ defined by

$$\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^n : y(t;\mathbf{w}) = y(t;0)\} = \left\{\mathbf{w} \in \mathbb{R}^n : Ce^{At}\mathbf{w} = 0\ \forall t > 0\right\}.$$

Deduce the desired equivalence after proving that

$$\mathcal{W} = \text{Image}\left(\left[C^T \mid A^TC^T \mid (A^T)^2C^T \mid \cdots \mid (A^T)^{n-1}C^T\right]\right)^{\perp}. \qquad (**)$$

(It may help to scour your lecture notes for a similar-looking equation proved in class.)

2. Consider the linearized satellite system with only tangential thrust. Taking $\rho = \omega = 1$, this is described by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Construct a feedback matrix F such that $A + BF$ has eigenvalues -2, -1, and $-\frac{1}{2} \pm \frac{1}{2}i$.

3. Linearizing the famous "broom-balancing problem" leads to a system $\dot{\mathbf{x}} = A\mathbf{x} + Bu$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{$*$}$$

(a) Show that system $(*)$ is controllable.

(b) Find a coordinate-transformation matrix P that puts $(*)$ into controllable canonical form.

(c) Find a feedback matrix F, for which all four eigenvalues of $A + BF$ equal -1.

(d) Show that if $y = x_3$ is the only state component we can measure, then $(*)$ is not observable. Give an example of two 0-input state trajectories that cannot be distinguished using this measurement; give a common-sense explanation based on the physical apparatus.

(e) Show that if $y = x_1$ is the only state component we can measure, then $(*)$ is observable.

(f) Using $y = x_1$ as the output, design an output-feedback controller that stabilizes the system. (Arrange for all eigenvalues of the error dynamics to equal -5.)

(g) Compare the output-feedback controller to the state-feedback controller numerically as follows. Plot, on the same axes, the function $t \mapsto x_1(t)$ corresponding to initial point $\mathbf{x}(0) = (1, 0, 0, 0)$ obtained using the 4×4 state-feedback system in (c) and the same function generated using the 8×8 output-feedback system in (f). Do the same for the the function $t \mapsto x_3(t)$. Then repeat this procedure (plotting both x_1 and x_3) for the initial point $\mathbf{x}(0) = (0, 0, 1, 0)$. In both cases, take the initial state estimate as $\mathbf{z}(0) = \mathbf{0}$.

4. For any compact set C in $\mathbb{R}^n$, one defines the *support function* $\sigma_C \colon \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\sigma_C(p) := \max\left\{p^T x \,:\, x \in C\right\}.$$

(a) Compute the support function for each of the following subsets of the plane:

$$\begin{aligned} C_1 &= \{(-1,-1),\ (-1,1),\ (1,-1),\ (1,1)\}, \\ C_2 &= \{(x,y) \,:\, |x| \le 1,\ |y| \le 1\}, \\ C_3 &= C_2 \cup \left\{(x,y) \,:\, (x-1)^2 + y^2 \le 1\right\}. \end{aligned}$$

(b) Suppose the set D is compact and convex, while the set C is merely compact. Prove: if $\sigma_C(p) \le \sigma_D(p)$ for all p in $\mathbb{R}^n$, then $C \subseteq D$. Give an example to show that this statement is false without the condition that D be convex.

5. Two closed convex sets $A, C \subseteq \mathbb{R}^n$ are given, and satisfy

$$\operatorname{int}(C) \ne \emptyset, \qquad A \cap \operatorname{int}(C) = \emptyset$$

Prove: If $p \in A \cap C$, there exists a nonzero vector w in $\mathbb{R}^n$ satisfying, simultaneously,

$$w \in N_A(p), \qquad -w \in N_C(p).$$

[Clue: Let $S = A - C = \{x - y \,:\, x \in A,\ y \in C\}$. Show that S is convex, and $0 \notin \operatorname{int}(S)$.]