M403(101) Solutions—Problem Set 4

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1. Statements (a) and (b) are logically equivalent, because each is the contrapositive of the other. It suffices to prove that either one is equivalent to (c).

Consider the set of initial vectors that the observer output cannot distinguish from $\mathbf{0}$, namely,

$$\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^n \,:\, \mathbf{y}(t;\mathbf{w}) = \mathbf{y}(t;\mathbf{0})\} = \left\{\mathbf{w} \in \mathbb{R}^n \,:\, Ce^{At}\mathbf{w} = 0 \ \forall t > 0\right\}.$$

Clearly $\mathcal{W}$ is a linear subspace of $\mathbb{R}^n$. Also, two initial vectors ξ and η generate the same observer trajectories, i.e., $\mathbf{y}(t;\xi) = y(t;\eta)$ for all $t > 0$, if and only if $\mathbf{w} \overset{\text{def}}{=} \eta - \xi$ lies in $\mathcal{W}$.

Standard uniqueness facts about ODE systems reveal that

$$\mathcal{W} = \left\{\mathbf{w} \in \mathbb{R}^n \,:\, \text{for some } T > 0, \text{ one has } \mathbf{y}(t;\mathbf{w}) = \mathbf{0} \ \forall t \in (0,T)\right\}.$$

Thus statements (a) and (b) simply assert that $\mathcal{W} = \{\mathbf{0}\}$. The desired equivalence with (c) says

$$\mathcal{W} = \{\mathbf{0}\} \quad \iff \quad \text{the pair } (A^T, C^T) \text{ is controllable.} \qquad (*)$$

Line $(*)$ follows immediately from a more general fact we prove below, namely,

$$\mathcal{W} = \text{Image}\left(\left[C^T \mid A^TC^T \mid (A^T)^2C^T \mid \cdots \mid (A^T)^{n-1}C^T\right]\right)^{\perp}. \qquad (**)$$

$(\supseteq)$: Choose any vector $\mathbf{w}$ in the right side of $(**)$. By definition, this means

$$\mathbf{w}^T\left[C^T \mid A^TC^T \mid (A^T)^2C^T \mid \cdots \mid (A^T)^{n-1}C^T\right] = \mathbf{0}^T.$$

Expanding this equation block by block reveals

$$\mathbf{0}^T = \mathbf{w}^T(A^T)^kC^T, \quad \text{i.e.,} \quad CA^k\mathbf{w} = 0, \qquad \forall k = 0,1,2,\ldots,n-1. \qquad (*)$$

Now the Cayley-Hamilton Theorem implies that $(*)$ actually holds for all $k \geq 0$. Therefore

$$\mathbf{0} = Ce^{At}\mathbf{w} = \left[C + \frac{t}{1!}CA + \frac{t^2}{2!}CA^2 + \frac{t^3}{3!}CA^3 + \cdots\right]\mathbf{w} \qquad \forall t > 0.$$

In particular, $\mathbf{y}(t;\mathbf{w}) = \mathbf{0}$ for all $t > 0$, so $\mathbf{w} \in \mathcal{W}$.

$(\subseteq)$: Pick any $\mathbf{w} \in \mathcal{W}$. This means

$$\mathbf{0} = \mathbf{y}(t;\mathbf{w}) = C\mathbf{x}(t;\mathbf{w}) = Ce^{At}\mathbf{w} \qquad \forall t > 0.$$

For any integer $k > 0$, taking k time derivatives in the identity above gives

$$\mathbf{0} = CA^ke^{At}\mathbf{w}, \qquad \forall t > 0.$$

In particular, taking limits as $t \to 0^+$ gives

$$CA^k\mathbf{w} = \mathbf{0}, \qquad \forall k = 0,1,2,\ldots.$$

Using only the first n identities in this infinite list, we have

$$\mathbf{0}^T = \mathbf{w}^T\left[C^T \mid A^TC^T \mid (A^T)^2C^T \mid \cdots \mid (A^T)^{n-1}C^T\right].$$

This shows that $\mathbf{w}$ lies in the right side of $(**)$. That equation is now completely proved.

To make the connection between $(**)$ and (c) completely explicit, recall that a general pair (A_0, B_0) is controllable if and only if $\text{Image}\,(\mathcal{C}_0)^{\perp} = \{0\}$ for their corresponding controllability matrix $\mathcal{C}_0$. Substituting $A_0 = A^T$ and $B_0 = C^T$ in this known fact, we have controllability of (A^T, C^T) if and only if

$$\{\mathbf{0}\} = \text{Image}\left(\left[C^T \mid A^TC^T \mid (A^T)^2C^T \mid \cdots \mid (A^T)^{n-1}C^T\right]\right)^{\perp}.$$

The right side is precisely $\mathcal{W}$, by $(**)$.

2. Direct approach: A polynomial with roots at the given λ-values is

$$q(\lambda) \overset{\text{def}}{=} (\lambda+1)(\lambda+2)(\lambda+\tfrac{1}{2}-\tfrac{1}{2}i)(\lambda+\tfrac{1}{2}+\tfrac{1}{2}i) = \lambda^4+4\lambda^3+\tfrac{11}{2}\lambda^2+\tfrac{7}{2}\lambda+1.$$

Meanwhile, if $F=[f_0\ f_1\ f_2\ f_3]$, then the characteristic polynomial of $A+BF$ is

$$\det(\lambda I-[A+BF]) = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -3 & \lambda & 0 & -2 \\ 0 & 0 & \lambda & -1 \\ -f_0 & 2-f_1 & -f_2 & \lambda-f_3 \end{vmatrix}$$
$$= \lambda^4 - f_3\lambda^3 + (1-2f_1-f_2)\lambda^2 + (3f_3-2f_0)\lambda + 3f_2.$$

These polynomials will coincide, hence have the same roots, if their coefficients match. Thus we must choose F so that

$$-f_3 = 4, \qquad 1-2f_1-f_2 = \frac{11}{2}, \qquad 3f_3-2f_0 = \frac{7}{2}, \qquad 3f_2 = 1.$$

It's not hard to deduce that $F = \begin{bmatrix} -\dfrac{31}{4} & -\dfrac{29}{12} & \dfrac{1}{3} & -4 \end{bmatrix}$.

Recipe Approach: The controllability matrix for this system is

$$C = \left[B \,|\, AB \,|\, A^2B \,|\, A^3B\right] = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 1 & 0 & -4 & 0 \end{bmatrix}.$$

It has full rank, so the indicated eigenvalue assignment is certainly possible.

The characteristic polynomial of A is (plug in $F=0$ above) $p(\lambda)=\lambda^4+\lambda^2$. The coefficients in p provide the entries for this matrix T:

$$T = \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ a_2 & a_3 & 1 & 0 \\ a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now, as suggested by theoretical work done in class, define

$$P = CT = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 1 & 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}.$$

Calculation shows that

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 1 & -2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 9 & 0 & 0 & 6 \end{bmatrix},$$

$$\text{so} \quad \widetilde{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \widetilde{B} = P^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We must choose $\widetilde{F}$ so that the characteristic polynomial of $\widetilde{A}+\widetilde{B}\widetilde{F}$ is $q(\cdot)$ (see above). That is, adding $\widetilde{B}\widetilde{F}$ to $\widetilde{A}$ must replace the bottom row with the negated coefficients of q:

$$\widetilde{F} = [\,-1 \quad -\tfrac{7}{2} \quad -\tfrac{11}{2} \quad -4\,] - [\,0 \quad 0 \quad -1 \quad 0\,] = \tfrac{1}{2}[\,-2 \quad -7 \quad -9 \quad -8\,].$$

For the original system, the desired feedback matrix is

$$\begin{aligned} F = \widetilde{F}P^{-1} &= \frac{1}{12}[\,-2 \quad -7 \quad -9 \quad -8\,]\begin{bmatrix} 0 & 1 & -2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 9 & 0 & 0 & 6 \end{bmatrix} \\ &= \frac{1}{12}[\,-93 \quad -29 \quad 4 \quad -48\,]. \end{aligned}$$

3. (a) The controllability matrix for the linear system $(*)$ is

$$\mathcal{C} = \big[B \mid AB \mid A^2B \mid A^3B\big] = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 1 & -1 & 2 & -3 \\ 0 & -1 & 1 & -3 \\ -1 & 1 & -3 & 4 \end{bmatrix}.$$

Hand or computer-assisted calculations show $\det(\mathcal{C}) = 1$, so $\mathcal{C}$ is nonsingular and the system is controllable.

(b) The characteristic polynomial of the system matrix A is

$$\det(sI - A) = \begin{vmatrix} s & -1 & 0 & 0 \\ 0 & s+1 & 1 & 0 \\ 0 & 0 & s & -1 \\ 0 & -1 & -2 & s \end{vmatrix} = s^4 + s^3 - 2s^2 - s.$$

In class we used the general notation $s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$; matching coefficients gives $a_0 = 0$, $a_1 = -1$, $a_2 = -2$, $a_3 = 1$. As shown in class, a suitable coordinate-change matrix is

$$\begin{aligned} P = \mathcal{C}\begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ a_2 & a_3 & 1 & 0 \\ a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & -1 & 2 \\ 1 & -1 & 2 & -3 \\ 0 & -1 & 1 & -3 \\ -1 & 1 & -3 & 4 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

We will later need to know the inverse of this matrix:

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Calculations involving P^{-1} confirm the predictions of the theory, namely,

$$\widetilde{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}, \qquad \widetilde{B} = P^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) Now $(s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1$, so we choose $\widetilde{F}$ to make

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix} = \widetilde{A} + \widetilde{B}\widetilde{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \widetilde{f}_0 & \widetilde{f}_1 + 1 & \widetilde{f}_2 + 2 & \widetilde{f}_3 - 1 \end{bmatrix}.$$

Hence $\widetilde{F} = [\,-1 \quad -5 \quad -8 \quad -3\,]$, and the matrix we want is

$$F = \widetilde{F}P^{-1} = [\,-1 \quad -5 \quad -8 \quad -3\,] \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = [\,1 \quad 5 \quad 9 \quad 8\,].$$

The matrix $A+BF$ is written out in part (g) below; the computer confirms that its eigenvalues are all -1.

(d) When the only state component we can see is x_3, our observation is $y = C\mathbf{x}$ with $C = [0\ 0\ 1\ 0]$. The matrix to test for observability of (A, C) is

$$\left[C^T \mid A^TC^T \mid (A^T)^2C^T \mid (A^T)^3C^T\,\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

The zero row here shows that the matrix fails to have full rank. A vector that annihilates this matrix is $\mathbf{w} = \widehat{\mathbf{e}}_1$, which corresponds to the position measurement $x_1 = \delta$. It's clear that knowing the entire time-history of the angle variable $\phi = x_3$ is not enough to uniquely determine the initial position of the cart. To be completely explicit, any real c provides a constant function $\mathbf{x}(t) = (c, 0, 0, 0)$ that is a system trajectory corresponding to $u(t) \equiv 0$, and there is no way to detect which c was chosen by looking at the function $t \mapsto C\mathbf{x}(t) = x_3(t) = 0$.

(e) When the only state component we can see is x_1, our observation is $y = C\mathbf{x}$ with $C = [1\ 0\ 0\ 0]$. The matrix to test for observability of (A, C) is

$$\left[C^T \mid A^TC^T \mid (A^T)^2C^T \mid (A^T)^3C^T\,\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

This is diagonal, with determinant 1, so it certainly has full row rank.

(f) The output feedback coefficient matrix L is to be chosen so that $\sigma(A+LC) = \{-5, -5, -5, -5\}$. This is equivalent to choosing L^T to assign $\sigma(A^T + C^TL^T)$, which is just like the problem we solved in part (d), except that it involves the matrix pair (A^T, C^T) instead of (A, B). Steps just like the ones shown above lead to $L = [\,-19 \quad -133 \quad 539 \quad 910\,]^T$.

(g) Implementing the state feedback scheme described in part (c) gives the system $\dot{\mathbf{x}} = (A+BF)\mathbf{x}$, i.e.,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 4 & 8 & 8 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Implementing the same state feedback strategy with the observer found in part (f) gives a coupled system for $\mathbf{x}$ and its estimate $\mathbf{z}$:

$$\dot{\mathbf{x}} = A\mathbf{x} + BF\mathbf{z}, \qquad \dot{\mathbf{z}} = A\mathbf{z} + BF\mathbf{z} + LC(\mathbf{z} - \mathbf{x}).$$

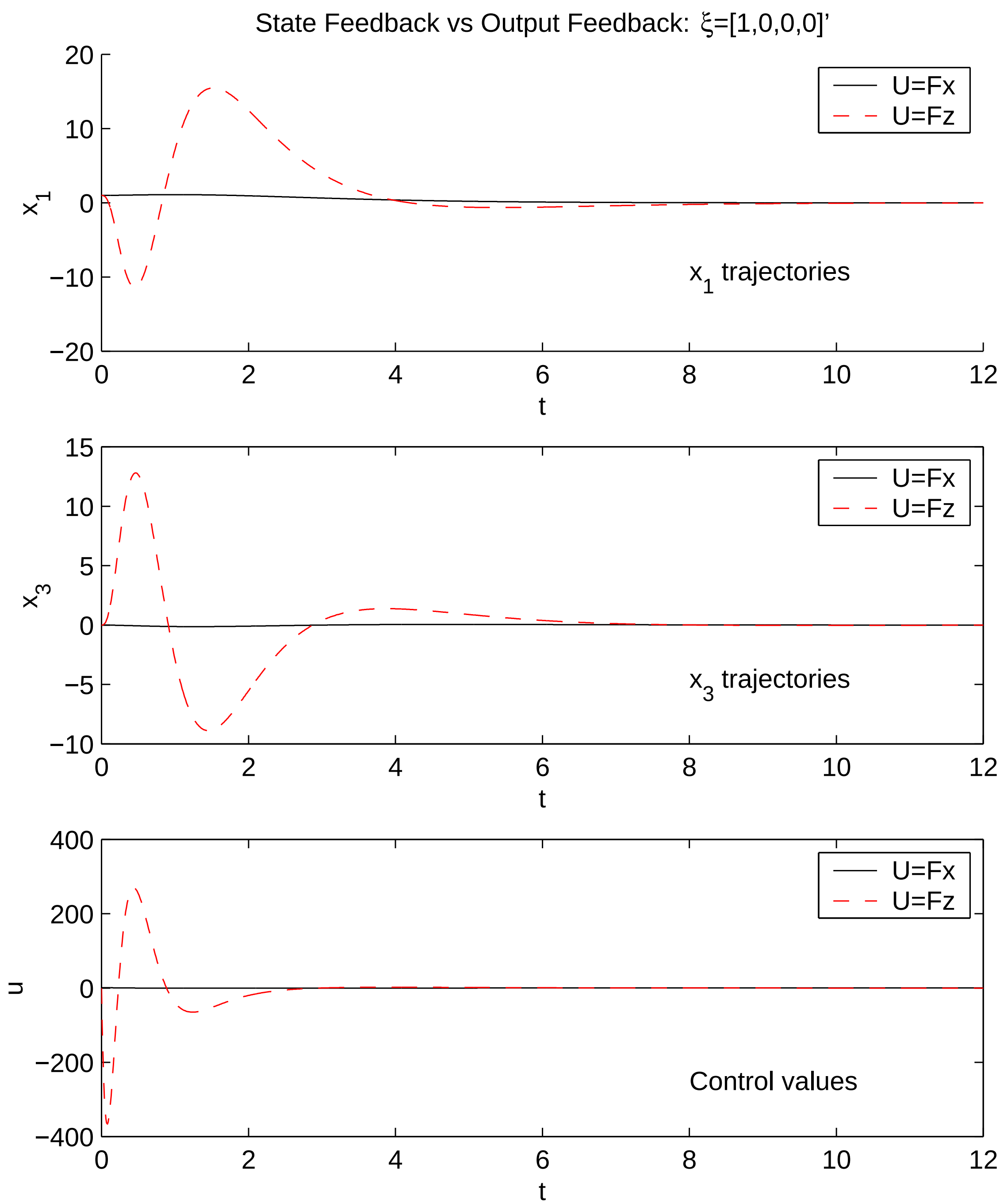
In block-matrix notation, this can be expressed as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}.$$

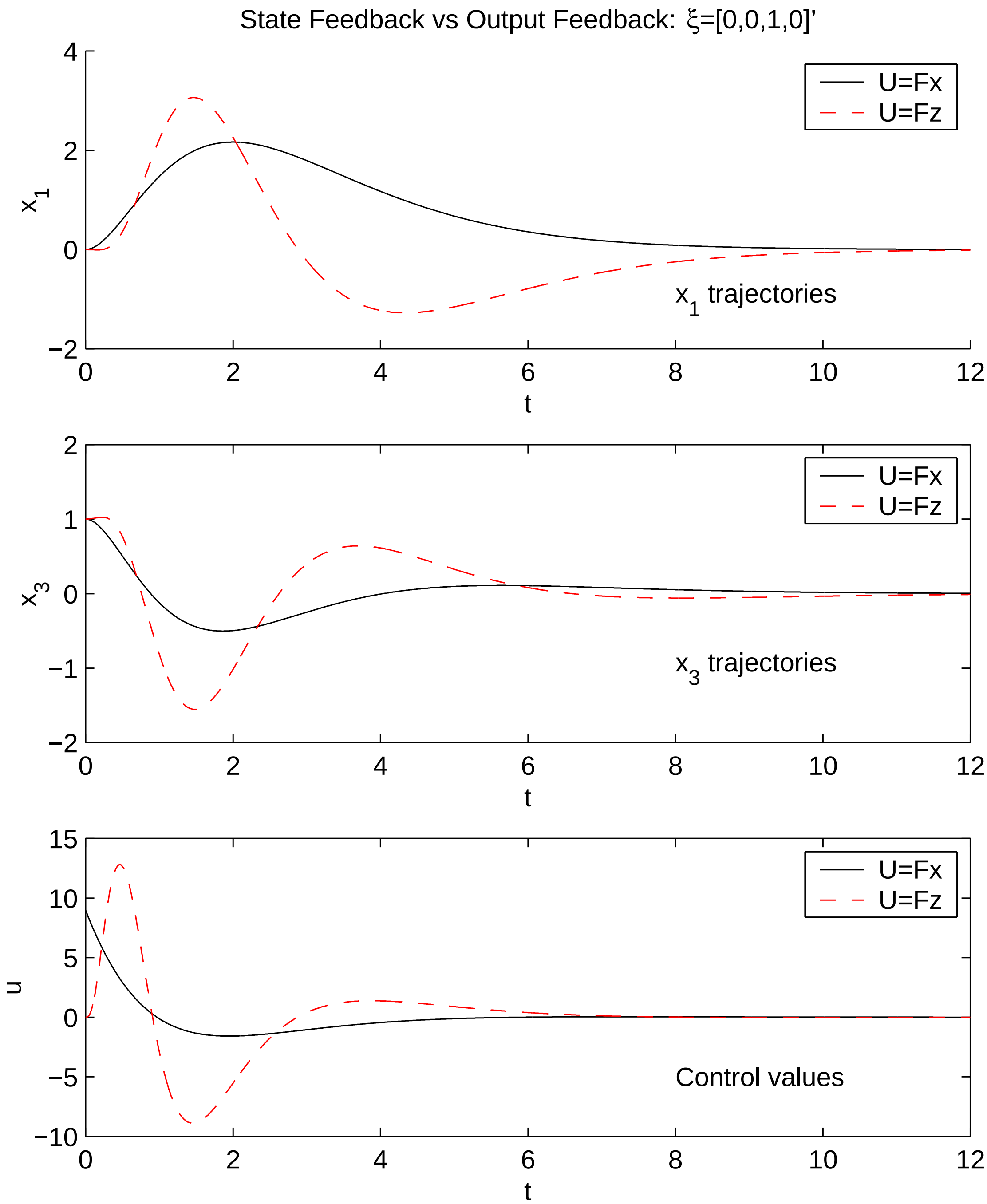
In fully explicit form, this is an 8×8 system of linear equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 5 & 9 & 8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & -5 & -9 & -8 \\ 19 & 0 & 0 & 0 & -19 & 1 & 0 & 0 \\ 133 & 0 & 0 & 0 & -132 & 4 & 8 & 8 \\ -539 & 0 & 0 & 0 & -539 & 0 & 0 & 1 \\ -910 & 0 & 0 & 0 & 909 & -4 & -7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}.$$

Running these two systems from the initial point $\widehat{\mathbf{e}}_1$ gives the state evolutions shown below:



Running these two systems from the initial point $\widehat{\mathbf{e}}_3$ gives the state evolutions shown below:



4. (a) Clearly $C_1 \subseteq C_2 \subseteq C_3$, so $\sigma_{C_1}(\mathbf{p}) \le \sigma_{C_2}(\mathbf{p}) \le \sigma_{C_3}(\mathbf{p})$ for each $\mathbf{p}$ in $\mathbb{R}^n$.

Claim: $\boxed{\sigma_{C_1}(p_1,p_2) = \sigma_{C_2}(p_1,p_2) = |p_1| + |p_2|.}$

To see this, note first that whenever $x = (x_1, x_2) \in C_2$, we have

$$\mathbf{p}^T\mathbf{x} = p_1x_1 + p_2x_2 \le |p_1x_1| + |p_2x_2| \le |p_1| + |p_2|,$$

so
$$\sigma_{C_1}(p_1,p_2) \le \sigma_{C_2}(p_1,p_2) \le |p_1| + |p_2|. \tag{1}$$

On the other hand, observe that for each real t one has $ts(t) = |t|$ for the function

$$s(t) \stackrel{\text{def}}{=} \begin{cases} -1, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases}$$

So for any $\mathbf{p} = (p_1, p_2)$, the point $(s(p_1), s(p_2))$ lies in C_1 and gives

$$(p_1, p_2) \bullet (s(p_1), s(p_2)) = |p_1| + |p_2|.$$

This shows that the upper bound on σ_{C_1} in (1) is attained, and gives $\sigma_{C_1}(p_1, p_2) = |p_1| + |p_2|$. Chaining this observation onto the left side of (1) establishes the claim.

Geometry: For a general compact set C in $\mathbb{R}^n$ and vector $\mathbf{p}$ in $\mathbb{R}^n$, the supremum defining $\sigma_C(\mathbf{p})$ is sure to be attained. If, in addition, the set C is convex and $\mathbf{p} \neq \mathbf{0}$, then the maximizer $\mathbf{x}$ must be a boundary point of C satisfying

$$\mathbf{p} \in N_C(\mathbf{x}).$$

To prove this, just expand the definition (using $\mathbf{x} \in C$):

$$\sigma_C(\mathbf{p}) = \mathbf{p}^T\mathbf{x} \iff \forall \mathbf{y} \in C,\ \mathbf{p}^T\mathbf{x} \geq \mathbf{p}^T\mathbf{y} \iff \forall \mathbf{y} \in C,\ \langle \mathbf{p}\,,\, \mathbf{y} - \mathbf{x} \rangle \leq 0.$$

Apply the geometric approach to set C_3. When $\mathbf{p} = (p_1, p_2)$ has $p_1 > 0$, the boundary point of C_3 where $\mathbf{p}$ is an outward normal lies on the arc of the circle $(x-1)^2 + y^2 = 1$ where the line to the centre point $(1, 0)$ is parallel to $\mathbf{p}$. This point is $\mathbf{x} = (1, 0) + \dfrac{\mathbf{p}}{\|\mathbf{p}\|}$, and it gives $\mathbf{p} \bullet \mathbf{x} = p_1 + \|\mathbf{p}\|$. When $p_1 \leq 0$ instead, the orientation of $\mathbf{p}$ focusses on the part of the boundary where C_3 has the same shape as C_2, and gives the same supremum as C_2. In summary,

$$\sigma_{C_3}(p_1, p_2) = \begin{cases} p_1 + \sqrt{p_1^2 + p_2^2}, & \text{if } p_1 > 0, \\ |p_1| + |p_2|, & \text{if } p_1 \leq 0. \end{cases}$$

(b) Proof by contraposition: Suppose C is *not* a subset of D. Then some point x in C must obey $x \notin D$. Since D is compact, it must be closed, and this means that x lies outside its closure. Since D is convex, the separation theorem proved in class implies that there must be some vector $\mathbf{p}$ in $\mathbb{R}^n$ such that

$$\forall \mathbf{d} \in D, \quad \mathbf{p}^T\mathbf{d} \leq \mathbf{p}^T\mathbf{x} - 1.$$

This implies that $\sigma_D(p) \leq \mathbf{p}^T\mathbf{x} - 1$. But $\mathbf{x} \in C$, so $\mathbf{p}^T\mathbf{x} \leq \sigma_C(\mathbf{p})$. Combining this with the previous inequality gives

$$\exists \mathbf{p} \in \mathbb{R}^n : \quad \sigma_D(\mathbf{p}) \leq \sigma_C(\mathbf{p}) - 1 < \sigma_C(\mathbf{p}).$$

That is, the statement, "$\sigma_C(\mathbf{p}) \leq \sigma_D(\mathbf{p})$ for all $\mathbf{p}$" is false, as required.

Part (a) supplies a ready-made counterexample: we showed there that $\sigma_{C_2} \leq \sigma_{C_1}$ everywhere (actually equality holds), although C_2 is clearly not covered by C_1. The point here is that the function σ_C is the same for all sets C that have the same convex envelope.

5. Let $S = A - C$. Whenever $s_0 = a_0 - c_0$ and $s_1 = a_1 - c_1$ are points of S, with $a_i \in A$ and $c_i \in C$ for $i = 0, 1$, and $t \in (0, 1)$, we have

$$s_t = (1-t)s_0 + ts_1 = [(1-t)a_0 + ta_1] - [(1-t)c_0 + tc_1] = a_t - c_t \in A - C = S.$$

Thus S is a convex set.

By shifting the coordinate axes if necessary, we may assume $p = 0$. Since $\mathrm{int}(C) \neq \emptyset$, there exists some $q \in \mathrm{int}(C)$. Since $0 \in C$, a lemma discussed in class ensures that $tq \in \mathrm{int}(C)$ for each $t \in (0, 1]$.

If $rq \in A - C$ for some $r > 0$, then a contradiction ensues. Indeed, this situation implies $rq = \widehat{a} - \widehat{c}$ for some $\widehat{a} \in A$ and $\widehat{c} \in C$, so $\widehat{a} - rq = \widehat{c} \in C$. The midpoint between this $\widehat{c}$ and the point $rq \in \mathrm{int}(C)$ must also belong to $\mathrm{int}(C)$: this point is $\frac{1}{2}\widehat{a} = \frac{1}{2}\widehat{a} + \frac{1}{2}0 \in A$. Thus $A \cap (\mathrm{int}\, C) \neq \emptyset$, a contradiction.

The previous paragraph implies that there is no $r > 0$ for which $rq \in A - C = S$. This implies that $p = 0$ lies on the boundary of the convex set S. It follows that some nonzero vector $w \in \mathbb{R}^n$ satisfies $w \in N_S(0)$, i.e.,

$$\forall a \in A,\ \forall c \in C, \quad \langle w\,,\, a - c\rangle \leq 0, \quad \text{i.e.,} \quad \langle w\,,\, a\rangle \leq \langle w\,,\, c\rangle\,.$$

Choosing $a = p$ and $c = p$ in turn, we deduce that

$$\forall a \in A,\ \forall c \in C, \quad \langle w\,,\, a\rangle \leq \langle w\,,\, p\rangle \leq \langle w\,,\, c\rangle\,, \quad \text{i.e.,} \quad \langle w\,,\, a - p\rangle \leq 0 \leq \langle w\,,\, c - p\rangle\,.$$

Splitting and rearranging these inequalities gives the defining characterizations of

$$w \in N_A(p), \qquad -w \in N_C(p).$$