Math 403 Problem Set 5

Due in class on Monday 29 October 2012

1. Consider this control system, in which $U = \{v \in \mathbb{R} : -1 \le v \le 1\}$:

$$\begin{aligned} \dot{x}_1 &= -x_1 - u \\ \dot{x}_2 &= -2x_2 - 2u, \qquad u \in [-1,1]. \end{aligned}$$

Find the attainable set $\mathcal{A}(T;0,U)$ for each $T > 0$. Show that every such set is contained in the ball $\mathbb{B}[\mathbf{0};\sqrt{2}]$, and that $\mathcal{A}(T;0,U)$ approaches a definite limit as $T \to \infty$. Contrast these properties with those of the attainable set for the rocket car discussed in class. Then identify the different characteristics of the respective system matrices A that account for these different behaviours.

2. Suppose the scalar-valued function $\widehat{u}$ is an extremal control for the single-input LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad -1 \le u(t) \le 1, \qquad \text{a.e. } t \in [0,1].$$

Prove: If there is some open subinterval (a,b) of $[0,1]$ such that $|\widehat{u}(t)| < 1$ for each $t \in (a,b)$, then the matrix pair (A,B) must fail to be controllable.

3. Imagine the rocket car fitted with a linear spring, so the dynamics become

$$\begin{aligned} \dot{x}_1 &= \quad x_2 \\ \dot{x}_2 &= -x_1 \quad + u, \qquad u \in [-1,1]. \end{aligned}$$

Calculate the attainable sets $\mathcal{A}(T;\mathbf{0},[-1,1])$ for each value of $T = \pi/2$, $T = \pi$, $T = 2\pi$, and $T = 3\pi/2$. For each of these given T-values, make a colour-coded two-part figure in which one panel shows representative directions for $\mathbf{p}(T)$ in the (p_1,p_2)-plane, while the other panel shows the points on the boundary of the attainable set for which these directions are outward normals. (We saw this sort of sketch in class, and there are many more on page 6 of the Rocket Car writeup on the course web page.)

Hints: Use geometry as well as algebra and analysis. There are many circular arcs to be found, and associated plane rotations with centres offset from the origin.

4. Consider the forced harmonic oscillator:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \qquad u \in [-1,1].$$

Find a nonlinear feedback control law that will drive any initial point $\mathbf{x}(0) = \vec{\xi}$ in $\mathbb{R}^2$ to the origin in minimum time. Draw a map of the (x_1,x_2)-plane showing the control value to use at each point.