M403(2012) Solutions—Problem Set 5

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1. The pre-Hamiltonian for this problem is

$$H(x,p,u) = -p_1 x_1 - 2p_2 x_2 - (p_1 + 2p_2)u.$$

An control-state pair $(u(\cdot), x(\cdot))$ is extremal iff some $p\colon [0,T] \xrightarrow[\text{PWS}]{} \mathbb{R}^2$ obeys

(a) $-\dot{p}_1 = H_{x_1} = -p_1$, which gives $p_1(t) = -Ae^t$ for some A,
$-\dot{p}_2 = H_{x_2} = -2p_2$, which gives $p_2(t) = -Be^{2t}$ for some B;

(c) $u(t)$ maximizes $v \mapsto -(p_1(t) + 2p_2(t))v$ over $|v| \le 1$, i.e.,

$$u(t) = \operatorname{Sgn}(-p_1(t) - 2p_2(t)) = \operatorname{Sgn}\left(Ae^t + 2Be^{2t}\right);$$

(d) $(p_1(T), p_2(T)) \neq (0,0)$, which implies that $(A,B) \neq (0,0)$.

Controls. Define $\phi(t) := Ae^t + 2Be^{2t}$, the "switching function". Note that

$$\phi(t) = 0 \iff 2Be^t = -A.$$

Since the left side is monotonic in t, this equation has at most one solution. Hence $u(\cdot)$ may have at most one switch, and must be piecewise constant with values ± 1.

Trajectories. On any interval where $u \equiv \sigma$ is constant, the system dynamics give

$$\begin{aligned} x_1(t) &= Ce^{-t} - \sigma, \\ x_2(t) &= De^{-2t} - \sigma \end{aligned} \tag{$*$}$$

for some constants C and D. These equations imply that for any t,

$$D\,(x_1 + \sigma)^2 = DC^2 e^{-2t} = C^2\,(x_2 + \sigma)\,. \tag{$**$}$$

Thus every constant control drives the system along a parabola in the (x_1, x_2)-plane. All the parabolas for a given control σ will have their vertex at the point $(-\sigma, -\sigma)$, and a complete range of curvatures can be realized as the arbitrary constants C and D vary.

Endpoints of Extremals. Fix a constant σ equal to either -1 or $+1$ and consider the one-parameter family of extremal controls that switches from σ to $-\sigma$ at time $\theta \in [0,T]$:

$$u_\theta(t) = \begin{cases} \sigma, & \text{if } 0 \le t < \theta, \\ -\sigma, & \text{if } \theta < t \le T, \end{cases} \qquad 0 \le \theta \le T.$$

On the interval $[0,\theta]$, the state evolves according to $(*)$ with $(0,0) = (x_1(0), x_2(0)) = (C - \sigma, D - \sigma)$. Thus

$$x_1(t) = \sigma e^{-t} - \sigma, \quad x_2(t) = \sigma e^{-2t} - \sigma, \qquad 0 \le t \le \theta.$$

On the interval $[\theta, T]$, we replace σ with $-\sigma$ in $(*)$ and enforce continuity at $t = \theta$. Thus the integration constants for this interval must obey

$$(e^{-\theta} - 1)\sigma = x_1(\theta) = Ce^{-\theta} + \sigma, \qquad (e^{-2\theta} - 1)\sigma = x_2(\theta) = De^{-2\theta} + \sigma.$$

We solve for $C = (1-2e^{\theta})\sigma$ and $D = (1-2e^{2\theta})\sigma$ and back-substitute to obtain

$$x_1(t) = \left[e^{-t} - 2e^{-(t-\theta)} + 1\right]\sigma, \quad x_2(t) = \left[e^{-2t} - 2e^{-2(t-\theta)} + 1\right]\sigma, \qquad \theta \le t \le T. \qquad (\dagger)$$

Isolating the exponential terms leads to a single equation in which θ does not appear:

$$2e^{-(t-\theta)} = \sigma x_1(t) - 1 - e^{-t}, \qquad 2\left[e^{-(t-\theta)}\right]^2 = \sigma x_2(t) - 1 - e^{-2t}.$$

$$\iff \qquad \left(\sigma x_2(t) - 1 - e^{-2t}\right) = \tfrac{1}{2}\left(\sigma x_1(t) - 1 - e^{-t}\right)^2$$

$$\iff \qquad x_2(t) = \sigma\left[1 + e^{-2t}\right] - \tfrac{1}{2}\sigma\left(x_1(t) - \sigma\left(1 + e^{-t}\right)\right)^2.$$

This relation is valid for each $t \ge \theta$, in particular, for $t = T$. It shows that the final state $(x_1(T), x_2(T))$ lies on a parabola in the (x_1, x_2)-plane. To determine the extent of this parabola, consider equation ($\dagger$) above, which implies that for either sign of σ and any switching time $\theta \in [0, T]$, we have

$$|x_1(T)| \le \left|1 + e^{-T} - 2e^{-(T-\theta)}\right|.$$

The function of θ inside the absolute value signs on the right is monotonically decreasing, with its maximum value of $1 - e^{-T}$ at $\theta = 0$ and its minimum value of $e^{-T} - 1$ at $\theta = T$. It follows that the range of allowable x_1-values is

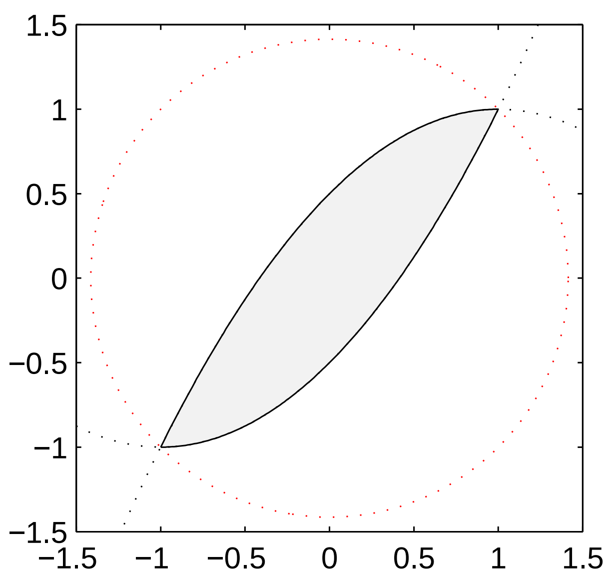
$$|x_1(T)| \le 1 - e^{-T}.$$

Together with the equation fixing the parabola above, this characterizes one half of the boundary of $\mathcal{A}(T; 0, U)$ when $\sigma = -1$ and the other half when $\sigma = +1$.

The set $\mathcal{A}(T; 0, U)$ may be found by plotting the parabolic arcs described above for $\sigma = \pm 1$ over the interval $-1 + e^{-T} \le x_1 \le 1 - e^{-T}$. It is a lens-shaped convex set which lies inside $\mathbb{B}[0; \sqrt{2}]$ for all $T > 0$ and increases to the limit

$$\left\{(x_1, x_2) \,:\, -1 + \tfrac{1}{2}(x+1)^2 \le x_2 \le 1 - \tfrac{1}{2}(x-1)^2\right\}.$$

This contrasts with the rocket car problem, in which the attainable sets grow without bound as time increases. The difference may be traced to the fact that in the current problem the matrix A in the dynamical system $\dot{x} = Ax + Bu$ has both its eigenvalues in the region $\Re e(\lambda) < 0$, so the system is asymptotically stable, whereas for the rocket car the eigenvalues of A are both 0, so the system is stable, but not asymptotically. The limiting attainable set is shown below.



2. Let $p\colon [0,1] \to \mathbb{R}^n$ be the costate that satisfies the definition of extremality for the given control $\widehat{u}$ and state evolution x. Then for almost every $t \in (a,b)$, the function below is maximized over the interval $[-1,1]$ by the choice $v = \widehat{u}(t)$:

$$v \mapsto p(t)^T Bv = (p(t)^T B)v.$$

This is a differentiable function, and since its maximizing point $v = \widehat{u}(t)$ is interior to the domain $[-1,1]$, its derivative must vanish at that point:

$$0 = \frac{d}{dv}\left((p(t)^T B)v\right)_{v=\widehat{u}(t)} = p(t)^T B, \qquad \forall t \in (a,b).$$

Now p is a nontrivial solution of $-\dot{p} = A^T p$, so $p(t) = e^{-(t-a)A^T} p(a)$; clearly $p(a) \neq 0$. So we have

$$0 = p(t)^T B = p(a)^T e^{-(t-a)A} B, \qquad \forall t \in [a,b].$$

Evaluating the k'th time derivative of the right side at $t = a$ (treating $k = 0$ as the original function) gives

$$0 = p(a)^T(-A)^k B, \qquad k = 0, 1, 2, \ldots.$$

Consequently $p(a)^T A^k B = 0$ for each $k = 0, 1, \ldots, n-1$, and this gives

$$0 = p(a)^T \left[\, B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B \,\right] = p(a)^T \mathcal{C}. \tag{‡}$$

In the single-input case, the controllability matrix $\mathcal{C}$ is square ($n \times n$), so equation (‡)—in which $p(a) \neq 0$—shows that $\mathcal{C}$ does not have full rank. Therefore the system is not controllable.

3. Here the preHamiltonian is

$$H(x_1, x_2, p_1, p_2, u) = p_1 x_2 - p_2 x_1 + p_2 u. \tag{1}$$

If the control $\widehat{u}(\cdot)$ is an extremal control on some interval, then the corresponding state trajectory $\mathbf{x}(\cdot)$ is related to some costate $\mathbf{p}(\cdot)$ for which

$$\widehat{u}(t) \in \arg\max_{v\in[-1,1]} H(\mathbf{x}(t), \mathbf{p}(t), v) = \arg\max_{v\in[-1,1]} \left\{ p_2(t) v \right\} = \begin{cases} \{-1\}, & \text{if } p_2(t) < 0, \\ [-1,1], & \text{if } p_2(t) = 0, \\ \{+1\}, & \text{if } p_2(t) > 0. \end{cases} \tag{2}$$

We get p_2 from solving the system

$$\begin{aligned} -\dot{p}_1(t) &= H_{x_1}(x_1(t), x_2(t), p_1(t), p_2(t), \widehat{u}(t)) = -p_2(t), \\ -\dot{p}_2(t) &= H_{x_2}(x_1(t), x_2(t), p_1(t), p_2(t), \widehat{u}(t)) = p_1(t). \end{aligned} \tag{3}$$

Differentiating the second equation and substituting from the first leads to

$$\ddot{p}_2(t) = -\dot{p}_1(t) = -p_2(t), \quad \text{i.e.,} \quad \ddot{p}_2(t) + p_2(t) = 0. \tag{4}$$

This is the harmonic oscillator equation, whose general solution can be expressed as

$$p_2(t) = R\sin(t - \delta), \tag{5}$$

for some $R \geq 0$ and $\delta \in [-\pi, \pi)$. From the original system, we deduce (with the same constants as in p_2)

$$p_1(t) = -\dot{p}_2(t) = -R\cos(t - \delta). \tag{6}$$

Extremality requires $\mathbf{p}(\cdot)$ to be nonvanishing, so we must have $R > 0$ above. The identity

$$p_1(t)^2 + p_2(t)^2 = R^2 \tag{7}$$

shows that the costate travels around a circle of radius R and centre $(0, 0)$ in the (p_1, p_2)-plane. Since $\mathbf{p}(\delta) = -R\widehat{\mathbf{e}}_1$ and $\dot{\mathbf{p}}(\delta) = R\widehat{\mathbf{e}}_2$, the motion is clockwise; it has constant angular speed 1 rad/sec. Only the sign of p_2 is relevant, so no generality is lost in taking $R = 1$.

The zeros of $p_2(\cdot)$ are periodically spaced, π radians apart, and one is at $t = \delta$. Let θ denote the smallest nonnegative zero of p_2, so $\theta = \delta$ if $\delta \in [0, \pi)$, but $\theta = \pi + \delta \in [0, \pi)$ if $\delta \in [-\pi, 0)$. Then, thanks to the identity $\sin(t \pm \pi) = -\sin(t)$, we can predict the form of $\widehat{u}$. There must be some choice of σ from the two-element set $\{-1, +1\}$ and θ from the interval $[0, \pi)$ for which $\widehat{u}$ is the restriction to $[0, T]$ of the following pattern:

$$\widehat{u}(t) = \begin{cases} \sigma, & \text{for } 0 < t < \theta, \\ -\sigma, & \text{for } \theta < t < \theta + \pi, \\ \sigma, & \text{for } \theta + \pi < t < \theta + 2\pi, \\ \vdots & \\ (-1)^{k+1}\sigma, & \text{for } \theta + k\pi < t < \theta + (k+1)\pi, \\ \vdots & \end{cases} \tag{8}$$

In detail, we use $\sigma = +1$ exactly when $\delta \in [-\pi, 0)$ and $\theta = \pi + \delta$; we use $\sigma = -1$ exactly when $\delta \in [0, \pi)$ and $\theta = \delta$.

On any open interval where $\widehat{u} \equiv \sigma$ (a constant), the system dynamics say

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \sigma, \tag{9}$$

giving $\ddot{x}_2 = -\dot{x}_1 = -x_2$, so $x_2(t) = \rho\cos(t - \phi)$ for some $\rho \geq 0$ and $\phi \in [-\pi, \pi)$; consequently $x_1(t) = \sigma - \dot{x}_2 = \sigma + \rho\sin(t - \phi)$. So as t advances through such an interval, the system state

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix} + \rho \begin{bmatrix} \sin(t - \phi) \\ \cos(t - \phi) \end{bmatrix} \tag{10}$$

travels clockwise along a circular arc whose centre is at the point $(\sigma, 0)$. The interval's length (increment of t) matches the angle subtended by the arc the state traverses.

Rotations. To rotate a given vector (x, y) *clockwise around the origin* through an angle ϕ, one multiplies (x, y) by the matrix

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}. \tag{11}$$

To use a given point (x_0, y_0) as the centre of rotation instead, one just shifts the origin: the original point (x, y) leads to a rotated image (x', y') according to

$$\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right),$$

$$\text{i.e.,} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}. \tag{12}$$

Case $T = \pi/2$. The complete catalogue of extremals can be generated by choosing arbitrary constants $\sigma \in \{-1, +1\}$ and $\theta \in [0, T]$ and taking

$$\widehat{u}(t) = \begin{cases} \sigma, & \text{for } 0 < t < \theta, \\ -\sigma, & \text{for } \theta < t < T. \end{cases} \tag{13}$$

On the first segment of duration θ, the initial state $(x, y) = (0, 0)$ gets rotated around $(x_0, y_0) = (\sigma, 0)$ clockwise through angle θ. It arrives at the point

$$\begin{bmatrix} x_1(\theta) \\ x_2(\theta) \end{bmatrix} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0-\sigma \\ 0-0 \end{bmatrix} = \begin{bmatrix} \sigma(1-\cos\theta) \\ \sigma\sin\theta \end{bmatrix}. \tag{14}$$

In the remaining segment, of duration $T - \theta$ (here $T = \pi/2$), the state $\mathbf{x}(\theta)$ just identified rotates clockwise through angle $T - \theta$ around the centre $(-\sigma, 0)$. The rotation formula above gives

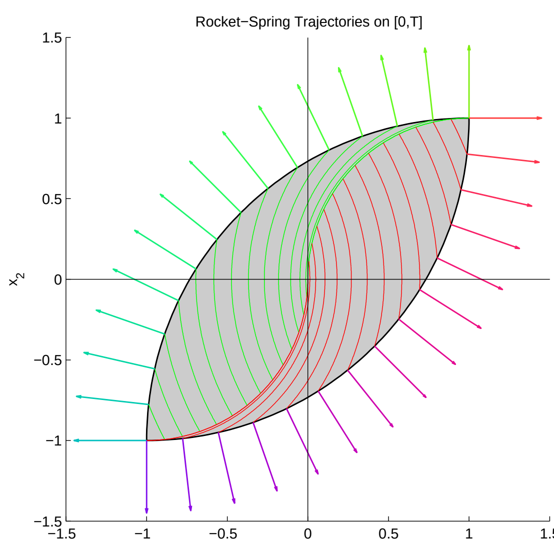
$$\begin{aligned} \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} &= \begin{bmatrix} -\sigma \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(T-\theta) & \sin(T-\theta) \\ -\sin(T-\theta) & \cos(T-\theta) \end{bmatrix} \begin{bmatrix} x_1(\theta)+\sigma \\ x_2(\theta)-0 \end{bmatrix} \\ &= \begin{bmatrix} -\sigma \\ 0 \end{bmatrix} + \begin{bmatrix} \sigma\cos(T-\theta)(2-\cos\theta) + \sigma\sin(T-\theta)\sin\theta \\ -\sigma\sin(T-\theta)(2-\cos\theta) + \sigma\cos(T-\theta)\sin\theta \end{bmatrix}. \end{aligned} \tag{15}$$

Substituting $T = \pi/2$ and using $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, we arrive at

$$\begin{bmatrix} x_1(\frac{\pi}{2}) \\ x_2(\frac{\pi}{2}) \end{bmatrix} = \sigma \begin{bmatrix} -1 + 2\sin\theta \\ 1 - 2\cos\theta \end{bmatrix} = \sigma \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} \right). \tag{16}$$

When $\sigma = +1$, each choice of $\theta \in [0, T]$ gives a point on the circle of radius 2 centred at $(-1, 1)$. Since $T = \pi/2$, these points fill the arc of a quarter-circle whose endpoints are at $(-1, -1)$ and $(1, 1)$. Choosing $\sigma = -1$ instead generates this quarter-circle's mirror image by changing the sign of every (vector) point. The set $\mathcal{A}$ is the convex region whose boundary is made of these two arcs.

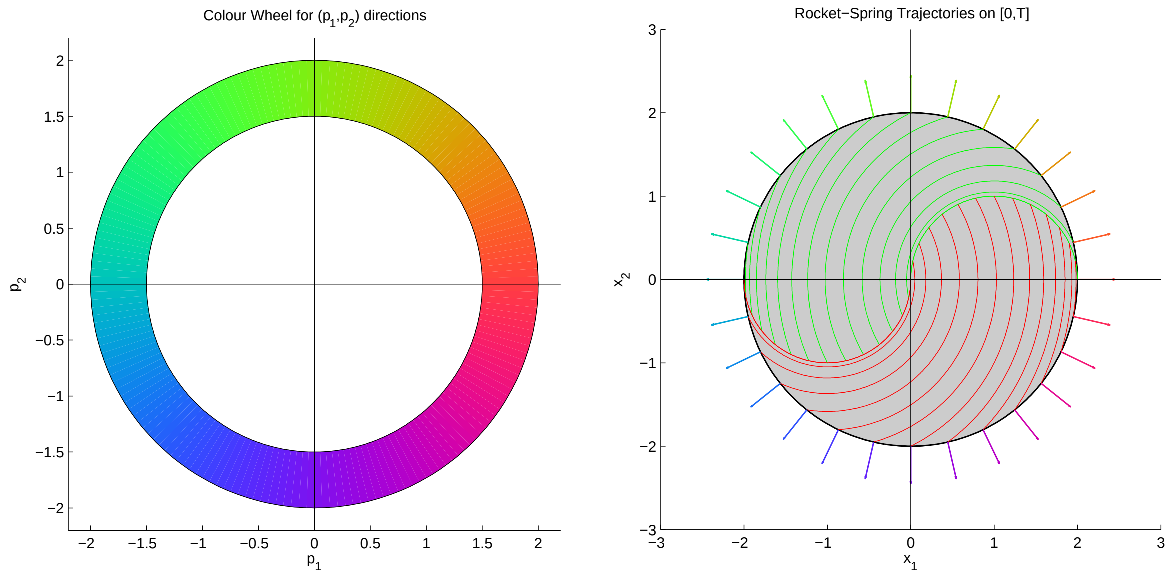
In the sketches below, state trajectories generated by $u = +1$ are circular arcs shown in green, while state trajectories along which $u = -1$ are circular arcs drawn in red. The state's motion along every arc is clockwise.



Case $T = \pi$. Since the first switching time θ in (13) obeys $\theta \geq 0$, we can be sure that $T = \pi < \theta + \pi$ in this case. Therefore any extremal control $\widehat{u}$ must have at most one switch. Hence the calculations given in the previous case can all be used without change until we reach line (15), where we must now substitute $T = \pi$. Noting that $\sin(\pi - \theta) = \sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$, we find

$$\begin{bmatrix} x_1(\pi) \\ x_2(\pi) \end{bmatrix} = -2\sigma \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \tag{17}$$

When $\sigma = +1$, each choice of $\theta \in [0, T]$ gives a point on the circle of radius 2 centred at $(0, 0)$. Since $T = \pi$, these points fill the arc of the lower half of this circle. Choosing $\sigma = -1$ instead generates this semicircle's mirror image by changing the sign of every (vector) point. The set $\mathcal{A}$ is the circular disk whose boundary is made of these two arcs.



Case $T = 2\pi$. Knowing that $0 \le \theta < \pi$ in (13), we now have $\pi \le \theta + \pi < 2\pi = T$. Therefore a two-switch control is inevitable:

$$\widehat{u}(t) = \begin{cases} \sigma, & \text{for } 0 < t < \theta, \\ -\sigma, & \text{for } \theta < t < \theta + \pi, \\ \sigma, & \text{for } \theta + \pi < t < T. \end{cases} \tag{18}$$

As in the cases above, we fix $\sigma \in \{-1, +1\}$ and $\theta \in [0, \pi)$ and the first interval of constant input ends with

$$\begin{bmatrix} x_1(\theta) \\ x_2(\theta) \end{bmatrix} = \begin{bmatrix} \sigma(1 - \cos\theta) \\ \sigma \sin\theta \end{bmatrix}. \tag{19}$$

This is followed by an interval of length π on which $\widehat{u}(\cdot) \equiv -\sigma$. The state arrives at a point we can calculate by clockwise rotation through π radians around the centre $(-\sigma, 0)$, namely,

$$\begin{aligned} \begin{bmatrix} x_1(\theta+\pi) \\ x_2(\theta+\pi) \end{bmatrix} &= \begin{bmatrix} -\sigma \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(\theta) + \sigma \\ x_2(\theta) - 0 \end{bmatrix} \\ &= \sigma \begin{bmatrix} -3 + \cos\theta \\ -\sin\theta \end{bmatrix}. \end{aligned} \tag{20}$$

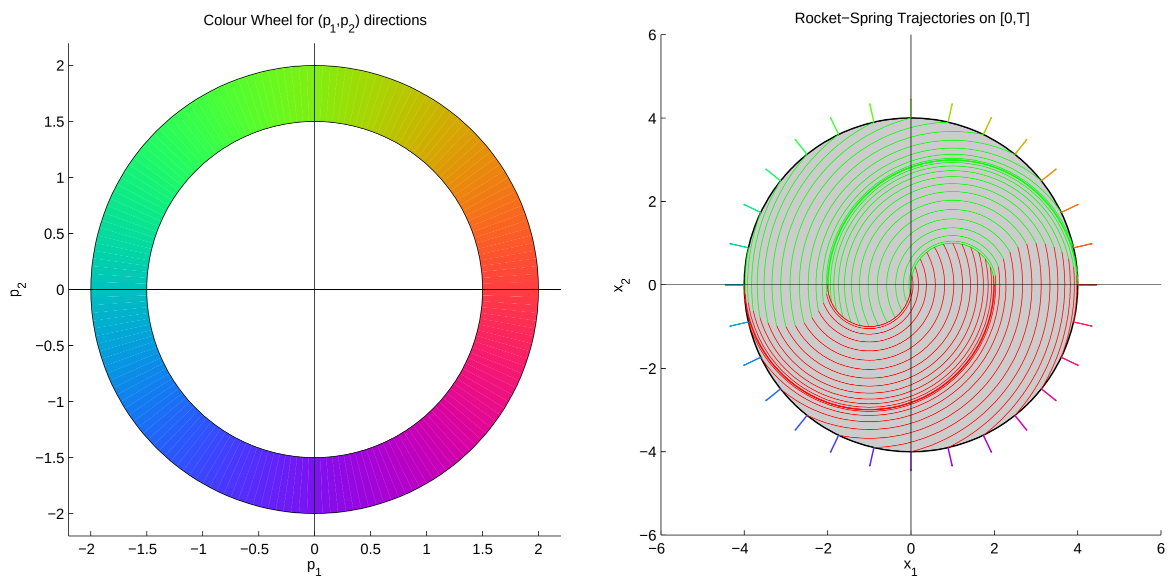
On the final segment, this state gets rotated clockwise through $\phi = T - (\theta + \pi) = T - \pi - \theta$ around the point $(\sigma, 0)$. It ends up at

$$\begin{aligned} \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} &= \begin{bmatrix} \sigma \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(T-\pi-\theta) & \sin(T-\pi-\theta) \\ -\sin(T-\pi-\theta) & \cos(T-\pi-\theta) \end{bmatrix} \begin{bmatrix} x_1(\theta+\pi) - \sigma \\ x_2(\theta+\pi) - 0 \end{bmatrix} \\ &= \sigma \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sigma \begin{bmatrix} -\cos(T-\theta) & -\sin(T-\theta) \\ \sin(T-\theta) & -\cos(T-\theta) \end{bmatrix} \begin{bmatrix} -4 + \cos\theta \\ -\sin\theta \end{bmatrix} \\ &= \sigma \begin{bmatrix} 1 - \cos(T-\theta)[-4 + \cos\theta] + \sin(T-\theta)\sin\theta \\ \sin(T-\theta)[-4+\cos\theta] + \cos(T-\theta)\sin\theta \end{bmatrix}. \end{aligned} \tag{21}$$

Substituting $T = 2\pi$, and noting the 2π-periodicity of the trig functions, we have

$$\begin{bmatrix} x_1(2\pi) \\ x_2(2\pi) \end{bmatrix} = 4\sigma \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \tag{22}$$

Much like the case when $T = \pi$, the set $\mathcal{A}$ is a circular disk centred at $(0,0)$. This time the disk's top semicircle is generated by $\sigma = +1$, and the bottom is generated by $\sigma = -1$.



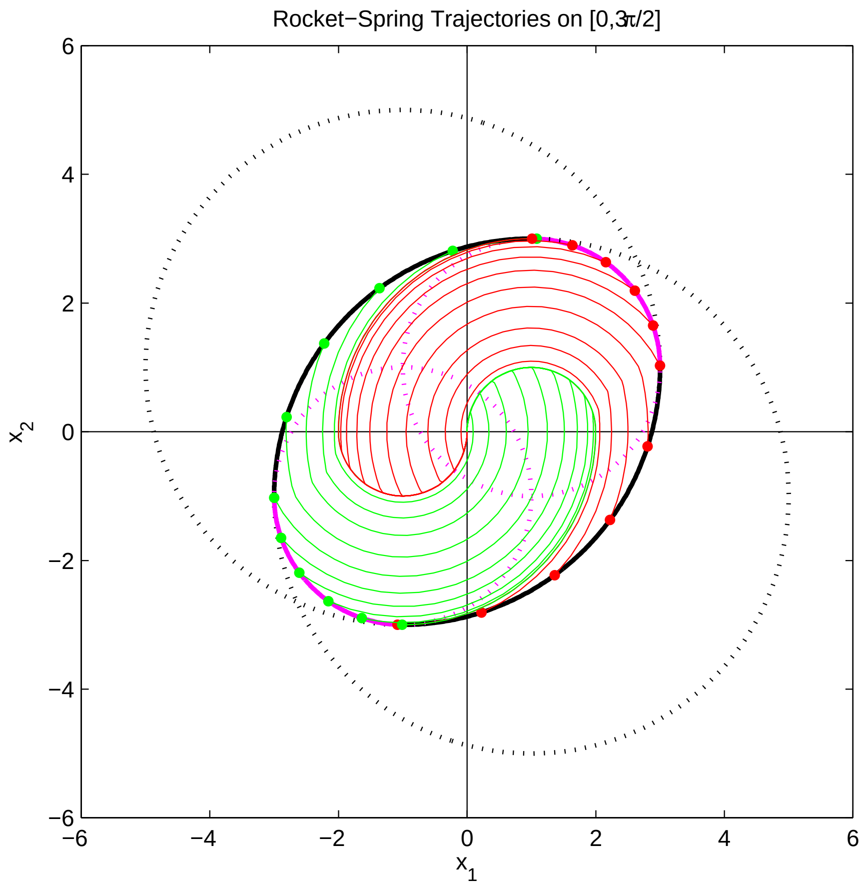
Case $T = 3\pi/2$. In this intermediate case between $T = \pi$ and $T = 2\pi$, there are two possible structures for an extremal control $\widehat{u}$. One possibility arises when $\theta + \pi < T = 3\pi/2$, i.e., when $\theta < \pi/2$. In this case the two-switch scenario just explored applies and we can re-use the work of the previous paragraph. Plugging $T = 3\pi/2$ into (21) leads to

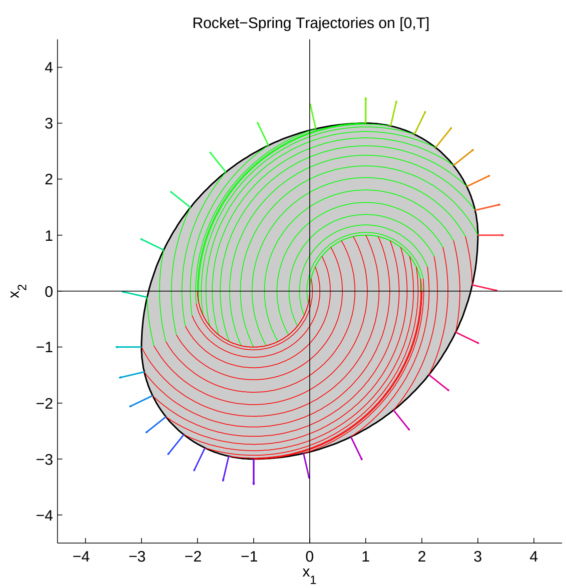
$$\begin{bmatrix} x_1(3\pi/2) \\ x_2(3\pi/2) \end{bmatrix} = \sigma \begin{bmatrix} 1 - 4\sin\theta \\ -1 + 4\cos\theta \end{bmatrix}, \qquad 0 \le \theta \le \frac{\pi}{2}. \tag{23}$$

The choices $\sigma = \pm 1$ generate two quarter-circles of radius 4. By contrast, when $\pi/2 < \theta < 3\pi/2$, there is only enough time for an extremal control to switch once. The hard work from the first case we studied becomes relevant again, but now we substitute $T = 3\pi/2$ into (15). The result is

$$\begin{bmatrix} x_1(3\pi/2) \\ x_2(3\pi/2) \end{bmatrix} = \sigma \begin{bmatrix} -1 - 2\sin\theta \\ -1 + 2\cos\theta \end{bmatrix}, \qquad \frac{\pi}{2} \le \theta \le \pi. \tag{24}$$

Again the choices $\sigma = \pm 1$ each generate a quarter-circle, except that these two arcs have radius 2 and centre points at $\pm(1,1)$.

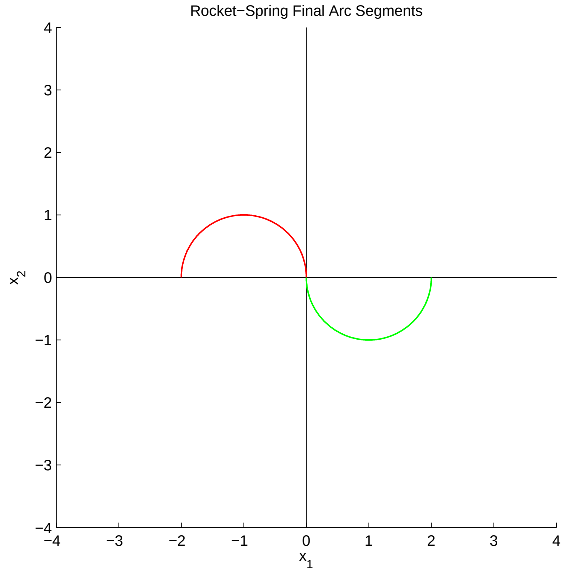




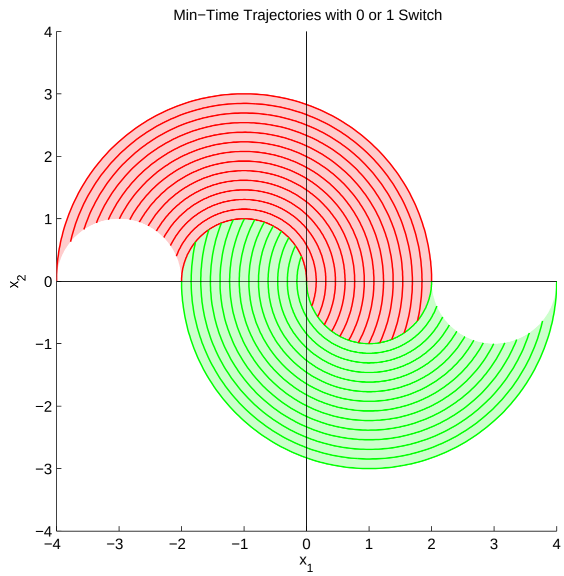
4. Any minimum-time trajectory that ends at the origin must be an extremal, and we found the general form of extremal arcs in Question 3. Each one is associated with a control that switches between piecewise constant segments at levels -1 and $+1$ according to

$$u(t) = \operatorname{sgn}\left(\sin(t-\delta)\right) \tag{1}$$

for some $\delta \in [-\pi, \pi)$. Thus a minimum-time trajectory to $(0,0)$ must arrive at the target by travelling clockwise along a circular arc. The circle $(x-1)^2 + y^2 = 1$ corresponds to $u = +1$, and the circle $(x+1)^2 + y^2 = 1$ corresponds to $u = -1$. But line (1) implies that no interval of constant control can have duration longer than π, so the final paths to the origin just described can be used exactly by starting states on the two semicircles shown below. As in Question 3, paths drawn in green have control $u = +1$, while the control produces the paths sketched in red.



Some initial states can get to the origin by switching onto one of the final arcs just identified by using the opposite constant control for some previous interval. But the maximum duration of this "previous interval" must be no longer than π, which would correspond to exactly one half-turn around the opposing circle. So the initial states that can transfer to the final green arc (where $u = 1$) are precisely the ones captured in the red-shaded region below. Some typical trajectories are provided. A symmetric analysis provides the green-shaded region.



Some initial states outside the shaded region can reach the shaded region in time not exceeding π by travelling along a constant-control arc and switching when they join a trajectory shown above. Allowing another π units of time captures the initial states shown below, along with the corresponding control strategies:

The pattern is probably clear by now. A boundary curve made of semicircles with radius 1 centred at points of the form $(2k+1,0)$, $k \in \mathbb{Z}$, describes the locus of switching points for time-optimal trajectories travelling toward the origin. Above the curve, the state travels clockwise around a circular arc centred at $(-1,0)$ by using control $u=-1$; below the curve, the state travels clockwise around a circular arc centred at $(+1,0)$ by using control $u=+1$. Each time the state hits the switching curve, it gets transferred to an arc of smaller radius. After a finite number of switches, it gets onto one of the arcs that terminate at the origin.

An extended picture of this situation appears below, with the switching curve shown in bold.