Math 403 Problem Set 6

Due in class on Wednesday 7 November 2012

1. Consider the dynamical system

$$\ddot{y}(t) = u(t), \quad 0 \le t \le 2; \qquad y(0) = 0,\ \dot{y}(0) = 0.$$

Call a piecewise continuous function $u\colon [0,2] \to \mathbb{R}$ a "control" if it obeys

$$|u(t)| \le 1, \qquad t \in [0,2].$$

(a) Find the control that maximizes the quantity $\dot{y}(2) - y(2)$.

(b) Among all controls which make $y(2) = 1$, find the one that maximizes $\dot{y}(2)$.

2. Consider the single-input, control-constrained system

$$\dot{x} = y, \quad \dot{y} = u, \quad |u| \le 1 \qquad \text{a.e. } t$$

and the target set

$$S = \{(x,y)\ :\ 0 \le y \le x+1,\ x \le 0\}.$$

For the problem of steering a given initial point in the (x,y)-plane to the target set S, carefully describe the set of initial points for which the minimum-time trajectory ends at a point where $y = x + 1$. Express the corresponding control strategies as functions of position in the (x,y)-plane.

3. Synthesize a feedback control strategy for minimum-time transfer to the origin, given

$$\dot{x}_1 = -3x_1 + 3u, \quad \dot{x}_2 = -x_2 + u, \quad |u| \le 1.$$

That is, use the Maximum Principle to help draw a "map" of the (x_1, x_2)-plane that specifies one u-value for each location. Trajectories generated by the corresponding controls should reach the origin in minimum time.

4. Consider the following calculus of variations problem involving a velocity constraint:

$$\min\left\{\int_0^2 (3\dot{x}(t) - 5x(t))\ dt\ :\ |\dot{x}(t) - x(t) - 1| \le 1,\ x(0) = 5\right\}.$$

Show how the choices $u = \dot{x} - x$ and $y(t) = \displaystyle\int_0^t (3\dot{x}(r) - 5x(r))\ dr$ can be used to transform this into a fixed-time optimal control problem with two-dimensional state space. Then find the minimizing piecewise smooth function $x(\cdot)$.