M403(2012) Solutions—Problem Set 6

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1. To put the dynamics into standard form, write $x_1 = y$, $x_2 = \dot{y}$. Then we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & x_1(0) &= 0, \\ \dot{x}_2(t) &= u(t), & x_2(0) &= 0, \\ u(t) &\in [-1,1]. \end{aligned}$$

We recognize the dynamics of the Rocket Car. The associated preHamiltonian is

$$H(t,x,p,u) = p_1 x_2 + p_2 u.$$

Solution I—Use Prior Work. (a) With our reformulation, we seek to minimize the function $\ell(x_1,x_2) = x_1 - x_2$ over all points in $\mathcal{A} = \mathcal{A}(2;0,U)$, the attainable set at time $T = 2$ for the rocket car. As discussed in class, a point η of the convex set $\mathcal{A}$ achieves the minimum if and only if

$$-\nabla\ell(\eta) \in N_{\mathcal{A}}(\eta).$$

Here $-\nabla\ell(\eta) = (-1,1)$ is the same at every point, and we know that outward normals to $\mathcal{A}$ provide the final values of the costate arc that the Maximum Principle associates with the corresponding state trajectory. So we seek an extremal in which $(p_1(2), p_2(2)) = (-1,1)$. The adjoint equation then gives $p_2(t) = t - 1$, and the maximum condition implies

$$\widehat{u}(t) = \operatorname{sgn}(t-1) = \begin{cases} -1, & \text{if } 0 \le t < 1, \\ +1, & \text{if } 1 < t \le 2. \end{cases}$$

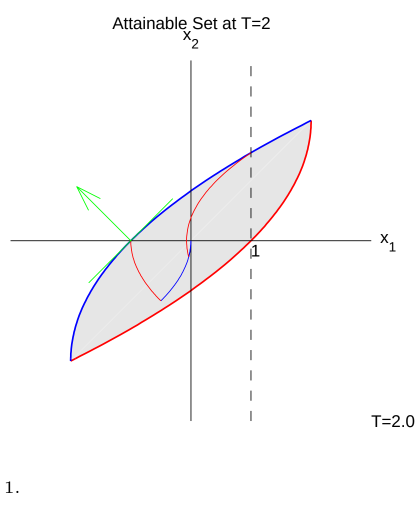
For $0 \le t \le 1$, the constant control $\widehat{u}(t) = -1$ gives the evolution

$$x_2(t) = -t, \quad x_1(t) = -\tfrac{1}{2}t^2; \quad \text{note} \quad x_1(1) = -\tfrac{1}{2},\ x_2(1) = -1.$$

Then, for $1 \le t \le 2$, the constant control $\widehat{u}(t) = +1$ (with the given starting point) gives

$$x_2(t) = -1 + (t-1) = t - 2,\ x_1(t) = \tfrac{1}{2}t^2 - 2t + 1.$$

The optimal endpoint is $(x_1(2), x_2(2)) = (-1, 0)$; the corresponding value is $x_2(2) - x_1(2) = 1$. Here is a sketch. The lens-shaped region is the attainable set $\mathcal{A}$ and the green arrow illustrates the vector $(-1,1)$.



(b) In the sketch above, the set of trajectory-endpoints that are allowed to compete for the maximum is formed by intersecting the set $\mathcal{A}$ with the vertical line $x_1 = 1$. We want to maximize x_2, so we want the top endpoint of this segment. In our study of the rocket car, we found a precise formula for the upper boundary of the set $\mathcal{A}$: it is the parabola [where $T = 2$]

$$x_1 = -\frac{1}{2}\left[T^2 - \frac{1}{2}(x_2 + T)^2\right], \qquad \text{i.e.,} \qquad 0 = x_1 + \frac{1}{2}\left[4 - \frac{1}{2}(x_2 + 2)^2\right] \stackrel{\text{def}}{=} \phi(x_1, x_2).$$

Substituting $x_1 = 1$ gives (the positive solution) $x_2 = 2(\sqrt{3} - 1)$. So this is the optimal final point for the trajectory we seek. At this final point, a vector normal to the boundary curve is

$$\nabla\phi(1, 2(\sqrt{3} - 1)) = (1, -\sqrt{3}).$$

This points down and to the right, so we must negate it to get an *outward* normal for $\mathcal{A}$ at the point of interest. This provides the value of $p(2)$ in the extremality conditions that describe the trajectory:

$$(p_1(2), p_2(2) = (-1, \sqrt{3}).$$

We deduce that $p_1(t) = -1$ is constant and that

$$p_2(t) = \sqrt{3} - 2 + t,$$

so, since $\widehat{u}(t) = \operatorname{sgn}(p_2(t))$,

$$\widehat{u}(t) = \begin{cases} -1, & \text{for } 0 < t < 2 - \sqrt{3}, \\ +1, & \text{for } 2 - \sqrt{3} < t < 2. \end{cases}$$

Integrating the system dynamics produces the second trajectory shown in the sketch above.

Solution II—Use the Maximum Principle Directly.

(a) With the dynamic reformulation above, our goal is to minimize $\ell(x_1(2), x_2(2))$, where $\ell(x_1, x_2) = x_1 - x_2$. If a control $\widehat{u}$ gives the minimum, it must be extremal. That is, the corresponding state evolution $x(\cdot)$ must come with some arc $p\colon [0, 2] \to \mathbb{R}^2$ satisfying the following conditions:

(AE) $-\dot{p}_1 = H_{x_1} = 0$, $-\dot{p}_2 = H_{x_2} = p_1$, with H evaluated at $(t, x(t), p(t), \widehat{u}(t))$.

This gives $p_1(t) = -m$ and $p_2(t) = mt + b$ for some constants m, b.

(GD) $\dot{x}_1 = H_{p_1} = x_2$, $\dot{x}_2 = H_{p_2} = \widehat{u}$, all evaluated along the trajectory as in (i).

This repeats the state equations. Further analysis appears below.

(MC) For almost all t, the choice $w = \widehat{u}(t)$ maximizes $H(t, x(t), p(t), w)$ over $w \in [-1, 1]$.

This gives $\widehat{u}(t) = \operatorname{sgn}(p_2(t)) = \operatorname{sgn}(mt + b)$.

(TC) $(-p_1(2), -p_2(2)) = \nabla\ell(x_1(2), x_2(2)) = (1, -1)$.

Since $p(2) = (-m, 2m+b)$, this amounts to $(m, -2m-b) = (1, -1)$, so $m = 1$ and $b = -1$.

From (TC), $p_2(t) = t - 1$. Hence, from (iii),

$$\widehat{u}(t) = \operatorname{sgn}(t - 1) = \begin{cases} -1, & \text{if } 0 \le t < 1, \\ +1, & \text{if } 1 < t \le 2. \end{cases}$$

In a strict interpretation, this completes the problem. But it's nice to see a little more: for $0 \le t \le 1$, the constant control $\widehat{u}(t) = -1$ gives the evolution

$$x_2(t) = -t, \quad x_1(t) = -\tfrac{1}{2}t^2; \quad \text{note} \quad x_1(1) = -\tfrac{1}{2},\ x_2(1) = -1.$$

Then, for $1 \le t \le 2$, the constant control $\widehat{u}(t) = +1$ (with the given starting point) gives

$$x_2(t) = -1 + (t-1) = t - 2,\ x_1(t) = \tfrac{1}{2}t^2 - 2t + 1.$$

The optimal endpoint is $(x_1(2), x_2(2)) = (-1, 0)$; the corresponding value is $x_2(2) - x_1(2) = 1$. These elements are in the sketch provided above.

(b) This time the final condition involves the target set $S = \{1\} \times \mathbb{R}$, and at every point of S [including the maximizing endpoint], the cone of outward normals is given by

$$N_S(x_1(2), x_2(2)) = \mathbb{R} \times \{0\}.$$

The conditions defining extremality of a control $\widehat{u}$ are all the same as in part (a), except that (TC) is replaced by

$$\begin{aligned} -(p_1(2), p_2(2)) &= \nabla\ell(x_1(2), x_2(2)) + N_S(x_1(2), x_2(2)), \\ \text{i.e.,} \qquad (m, -2m - b) &= (1, -1) + \mathbb{R} \times \{0\} \\ \text{i.e.,} \qquad m - 1 \in \mathbb{R}, &\quad -2m - b + 1 \in \{0\}. \end{aligned}$$

The first of these inclusions gives no useful information at all about m, but the second gives $b = 1 - 2m$, so that $p_2(t) = mt + 1 - 2m$. What information can we extract about the switching strategy for $\widehat{u}$? Notice that a constant control $\widehat{u} \equiv \sigma$ ($\sigma = \pm 1$) cannot hope to solve the problem, since it produces a trajectory that violates the endpoint constraint:

$$\widehat{u}(t) \equiv \sigma \text{ on } [0,2] \implies x(t) = (\tfrac{1}{2}\sigma t^2, \sigma t) \text{ on } [0,2] \implies x_1(2) = 2\sigma \ne 1.$$

So the function p_2 must have a sign change somewhere in the interval $(0, 2)$. Calculation shows $p_2(\tau) = 0$ iff $\tau = 2 - 1/m$, so we must have

$$0 < \tau < 2, \quad \text{i.e.,} \quad -2 < -\frac{1}{m} < 0, \quad \text{i.e.,} \quad m > \frac{1}{2}.$$

This implies that p_2 has positive slope, so it starts negative and finishes positive. The optimal control must have the form below for some τ:

$$\widehat{u}(t) = \begin{cases} -1, & \text{if } 0 \le t < \tau, \\ +1, & \text{if } \tau < t \le 2. \end{cases} \tag{$*$}$$

To find τ, we just integrate the dynamics and enforce the constraint. On the initial segment, $\widehat{u}(t) = -1$ gives the evolution

$$x_2(t) = -t, \quad x_1(t) = -\tfrac{1}{2}t^2; \quad \text{note} \quad x_1(\tau) = -\tfrac{1}{2}\tau^2,\ x_2(\tau) = -\tau.$$

On the final segment, $\widehat{u}(t) = +1$ and

$$\begin{aligned} x_2(t) &= -\tau + (t - \tau) = t - 2\tau, \\ x_1(t) &= -\tfrac{1}{2}\tau^2 - \tau(t - \tau) + \tfrac{1}{2}(t - \tau)^2 = \tau^2 - 2t\tau + \tfrac{1}{2}t^2. \end{aligned}$$

The trajectory finishes in the target line S if and only if

$$0 = x_1(2) - 1 = \tau^2 - 4\tau + 1 = \tau^2 - 4\tau + 4 - 3 = (\tau - 2)^2 - 3,$$

i.e., $\tau = 2 \pm \sqrt{3}$. Only the choice $\tau = 2 - \sqrt{3}$ lies in the interval $(0, 2)$, so this is the switching time we want: using it in $(*)$ completes the determination of $\widehat{u}$. The corresponding maximum value is

$$x_2(2) = \Big[2 - 2\tau\Big]_{\tau = 2 - \sqrt{3}} = 2(\sqrt{3} - 1).$$

2. The sketch supplied at the end of this writeup summarizes the answer. Here's how to derive it.

The conditions of the Maximum Principle involve $H((x,y),(p,q),u) = py + qu$, and stipulate that a time-optimal control $\widehat{u}$ that steers some initial point to S in minimum-time T must have corresponding state trajectory (x,y) and costate arc (p,q) satisfying the given system dynamics and

(a) $-\dot{p} = H_x = 0$, $-\dot{q} = H_y = p$, so p is constant and $q(t) = b - pt$ for some b.

(c) $\widehat{u}(t) \in \arg\max\{q(t)v \,:\, v \in [-1,1]\} = \operatorname{sgn}(q(t)) = \operatorname{sgn}(b - pt)$,

(d) $-(p(T), q(T)) \in N_S(x(T), y(T))$.

If the final point lies on a the face of S where $y = x+1$ and $-1 < x < 0$, then the outward normals to S all have the same direction as $(-1,1)$. Since the adjoint functions are scale-invariant, we may insist that $(-p, pT - b) = (-1,1)$, giving $p = 1$ and $b = T - 1$. Hence $q(t) = T - 1 - t$. So $q(t)$ is strictly decreasing, and changes sign from positive to negative when $t = T - 1$. The corresponding control has the form

$$\widehat{u}(t) = \begin{cases} 1, & \text{if } 0 \le t < T-1, \\ -1, & \text{if } T-1 < t \le T. \end{cases}$$

For a final point (x_0, y_0) obeying $y_0 = x_0 + 1$ the final trajectory obeys $x = K - \frac{1}{2}y^2$ with $K = x_0 + \frac{1}{2}y_0^2$, i.e.,

$$x = x_0 + \tfrac{1}{2}(y_0^2 - y^2).$$

The extreme cases $(x_0, y_0) = (-1, 0)$ and $(x_0, y_0) = (0, 1)$ give parabolas $x = -1 - \frac{1}{2}y^2$ and $x = \frac{1}{2} - \frac{1}{2}y^2$, respectively. Since the trajectories' displacement along the y-axis has unit speed, the switching point (x,y) must obey where $y = 1 + y_0$ and therefore

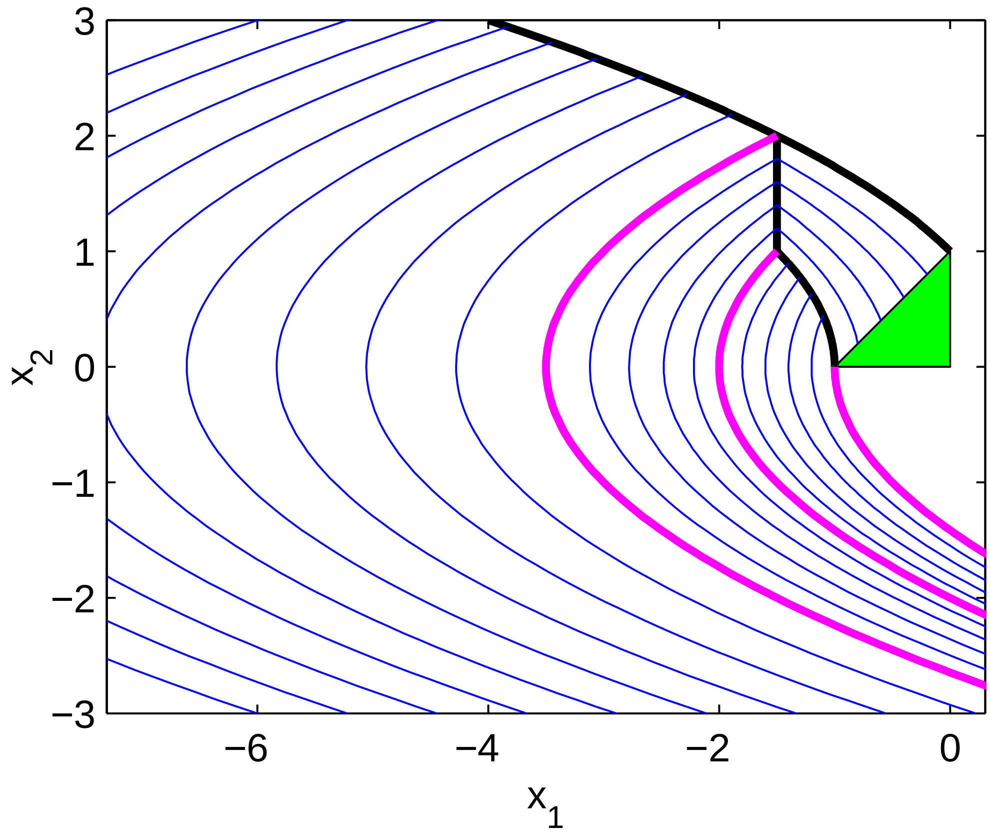
$$x = x_0 + \tfrac{1}{2}\left[y_0^2 - (y_0+1)^2\right] = x_0 + \tfrac{1}{2}\left[-1 - 2y_0\right] = x_0 + \tfrac{1}{2}\left[-1 - 2(1 + x_0)\right] = -\frac{3}{2}.$$

So the switching locus is the segment of $x = -3/2$ which runs between $y = 1$ and $y = 2$, its intersection points with the extreme parabolas already found.

If the final point lies at the vertex $(-1, 0)$ of S, there is a big range of choices for the possible normal to S. One points straight down, leading via (d) to $(0,-1) = -(p(T), q(T)) = (-p, pT - b)$. This gives $p = 0$ and $b = 1$, so $\widehat{u}(t) = 1$ always and the system cruises to $(-1,0)$ along $x = -1 + \frac{1}{2}y^2$. The others all point to the left, having form $(-1, k)$ for some $k \le 1$. For these condition (d) gives $(-1, k) = (-p, pT - b)$ so $p = 1$ and $b = T - k$, leading to $q(t) = T - t - k$. Each $k \le 1$ produces a different decreasing function; the sign changes at time $T - k$. The corresponding family of controls (one for each k) describes a band of trajectories that follow $u \equiv 1$ until they switch onto a short arc of the parabola $x = -1 - \frac{1}{2}y^2$ for their final trip to the target.

If the final point lies at the vertex $(0,1)$ of S, there are many possible normals, so there may be many corresponding trajectories. The normal $(1,0)$ leads to $(1,0) = (-p, pT - b)$ with $p = -1$ and $b = -T$, so $q(t) = -T - t$ is everywhere negative and the system follows $u \equiv -1$ forever. The normals $(r, 1)$ where $r \ge -1$ lead to $(r,1) = (-p, pT - b)$, i.e., $p = -r$ and $b = -rT - 1$, i.e., $q(t) = -rT - 1 + rt$. Whenever $r \ge 0$ we have $q(t) = -1 + r(t - T) < 0$ so the same trajectory is produced. But for $-1 \le r < 0$, q is strictly decreasing and may start positive. Its sign change occurs when $t = T + 1/r < T - 1$, which corresponds to a point on the parabola $x = \frac{1}{2} - \frac{1}{2}y^2$ above and to the left of the line segment found earlier.

Key features identified above are shown in this sketch:



On parabolic arcs opening to the right, the control obeys $u = 1$. In the small region adjacent to the hypotenuse of S, the time-optimal control is $u = -1$.

3. Here the preHamiltonian is

$$H(\mathbf{x},\mathbf{p},u) = p_1\,(-3x_1 + 3u) + p_2\,(-x_2 + u) = -3x_1p_1 - x_2p_2 + (3p_1 + p_2)u.$$

The extremality conditions relating an optimal control-state pair $\widehat{u}(\cdot)$, $\mathbf{x}(\cdot)$ to its costate $\mathbf{p}(\cdot)$ say

(a) $-\dot{p}_1(t) = H_{x_1}(\mathbf{x}(t),\mathbf{p}(t),\widehat{u}(t)) = -3p_1(t)$, so $p_1(t) = Ke^{3t}$ for some $K \in \mathbb{R}$;
$-\dot{p}_2(t) = H_{x_2}(\mathbf{x}(t),\mathbf{p}(t),\widehat{u}(t)) = -p_2(t)$, so $p_2(t) = Ce^{t}$ for some $C \in \mathbb{R}$.

(b) $\dot{x}_1(t) = H_{p_1}(\mathbf{x}(t),\mathbf{p}(t),\widehat{u}(t)) = -3x_1(t) + 3\widehat{u}(t)$,
$\dot{x}_2(t) = H_{p_2}(\mathbf{x}(t),\mathbf{p}(t),\widehat{u}(t)) = -x_2(t) + \widehat{u}(t)$,

(c) $\widehat{u}(t) \in \arg\max\{[3p_1(t) + p_2(t)]v \,:\, -1 \le v \le 1\}$, so $\widehat{u}(t) \in \mathrm{Sgn}(3p_1(t) + p_2(t))$ a.e..

(d) $-\mathbf{p}(T) \in N_{\{\mathbf{0}\}}(\mathbf{0}) = \mathbb{R}^2$, $\mathbf{p}(T) \ne \mathbf{0}$. This forces $K^2 + C^2 \ne 0$ in (a), but otherwise gives no useful information.

Introduce $\phi(t) = 3p_1(t) + p_2(t) = 3Ke^{3t} + Ce^{t}$, so $\widehat{u}(t) \in \mathrm{Sgn}\,\phi(t)$ in (c). Notice that

$$\phi(t) = 0 \iff 3Ke^{3t} = -Ce^{t} \iff 3Ke^{2t} = -C.$$

If $K \ne 0$, this has at most one solution for t. Case $K = 0$ is possible, but it forces $C \ne 0$ by (d), and $\phi(t) = Ce^{t}$ is then a function with no zeros at all. Thus we deduce that the control function $\widehat{u}$ is piecewise constant at level -1 or $+1$, with at most one jump.

On any open interval (a,b) where $\widehat{u}$ is constant, standard ODE-solving work reveals

$$\frac{d}{dt}[x_1-\widehat{u}] = \dot{x}_1 = -3[x_1-\widehat{u}] \implies x_1(t) = \widehat{u} + [x_1(a)-\widehat{u}]e^{-3(t-a)},$$
$$\frac{d}{dt}[x_2-\widehat{u}] = \dot{x}_2 = -[x_2-\widehat{u}] \implies x_2(t) = \widehat{u} + [x_2(a)-\widehat{u}]e^{-(t-a)}.$$

Both x_1 and x_2 are monotonic functions of t, converging to $\widehat{u}$ as $t\to\infty$. Eliminating t between the equations above gives

$$\left(\frac{x_1(t)-\widehat{u}}{x_1(a)-\widehat{u}}\right) = \left(\frac{x_2(t)-\widehat{u}}{x_2(a)-\widehat{u}}\right)^3. \qquad (**)$$

Thus the system state follows a cubic path in the (x_1,x_2)-plane, moving toward the point $(\widehat{u},\widehat{u})$. Some trajectories are shown in Figure 1.

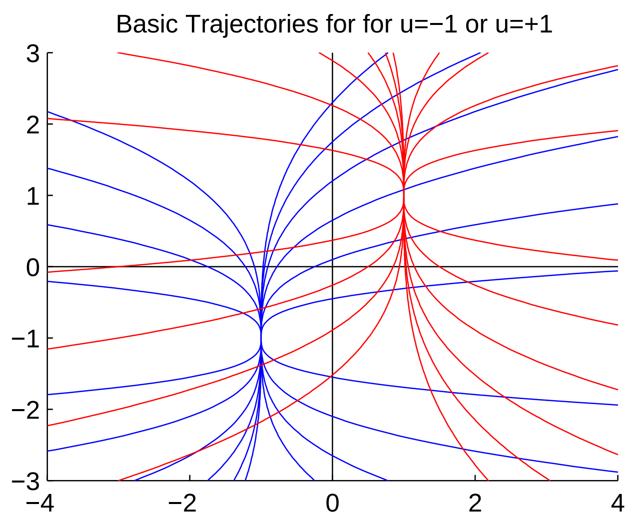


Fig. 1: Trajectories with $u=+1$ are red; trajectories for $u=-1$ are blue.

Switching Curve. A trajectory along which $\widehat{u}$ is constant obeys $\mathbf{x}(t)=\mathbf{0}$ for some t iff [see $(**)$]

$$\left(\frac{0-\widehat{u}}{x_1(a)-\widehat{u}}\right)^{-1} = \left(\frac{0-\widehat{u}}{x_2(a)-\widehat{u}}\right)^{-3}, \quad \text{i.e.,} \quad (x_1(a)-\widehat{u}) = (x_2(a)-\widehat{u})^3/\widehat{u}^2.$$

When $\widehat{u}\equiv +1$, this reveals a curve along which the state can ride to the origin:

$$S_+: \qquad x_1 = (x_2-1)^3+1, \quad x_2 \le 0.$$

When $\widehat{u}\equiv -1$, the state can ride to the origin along this curve in the first quadrant:

$$S_-: \qquad x_1 = (x_2+1)^3-1, \quad x_2 \ge 0.$$

Time-Optimal Synthesis. Let $\psi(y) = \begin{cases} (y-1)^3+1, & \text{if } y\le 0,\\ (y+1)^3-1, & \text{if } y>0.\end{cases}$

Then the best control strategy is $U(x_1,x_2) = \begin{cases} +1, & \text{if } x_1 > \psi(x_2), \\ -1, & \text{if } x_1 < \psi(x_2), \\ +1, & \text{if } x_1 = \psi(x_2) \text{ with } x_2 < 0, \\ -1, & \text{if } x_1 = \psi(x_2) \text{ with } x_2 = 0, \\ 0, & \text{if } x_1 = 0 = x_2. \end{cases}$

A sketch of the optimal feedback synthesis appears below. The curve $x = \psi(y)$ divides the plane into two symmetric pieces. Above the curve, the control $U = -1$ drives the state downward on a cubic curve with vertex $(-1,-1)$. The state hits the switching curve before reaching that vertex, and rides to the origin along $x = \psi(y)$ using the constant control $U = 1$. A symmetric description applies below the curve.

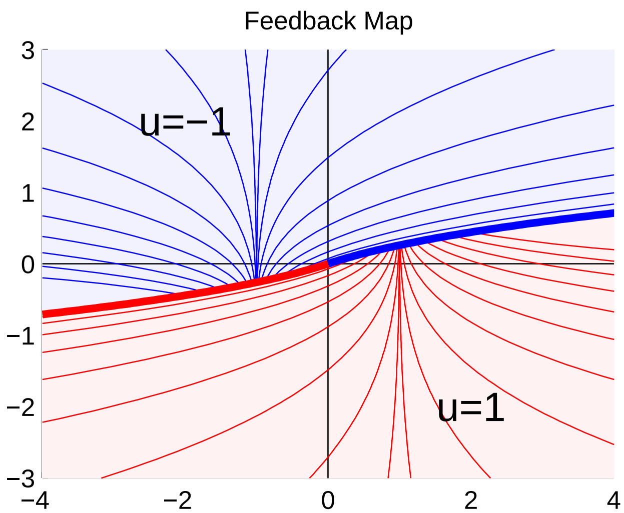


Fig. 2: Feedback synthesis.

4. Defining $u = \dot{x} - x$ and $y(t) = \displaystyle\int_0^t (3\dot{x}(r) - 5x(r))\,dr$, and then changing names to $x_1 = x$, $x_2 = y$ leads to the equivalent problem

$$\begin{aligned} \text{minimize}\quad & x_2(2) \\ \text{subject to}\quad & \dot{x}_1 = x_1 + u, && x_1(0) = 5, \\ & \dot{x}_2 = -2x_1 + 3u, && x_2(0) = 0, \\ & u \in [0,2]. \end{aligned}$$

This is a fixed-interval problem of a standard form. Its endpoint cost function is $\ell(x,y) = y$; the pre-Hamiltonian is $H(x,p,u) = p_1x_1 - 2p_2x_1 + (p_1 + 3p_2)u$. An optimal control $\widehat{u}$ must satisfy the usual conditions of the maximum principle, together with the transversality condition

(d) $(-p_1(2), -p_2(2)) = \nabla\ell(x(2)) = (0,1)$.

The adjoint equations say

(a) $-\dot{p}_1 = H_{x_1} = p_1 - 2p_2, \qquad -\dot{p}_2 = H_{x_2} = 0.$

The second of these implies that p_2 is constant, with the value $p_2 \equiv -1$ from (d); the first then gives $p_1 = Ae^{-t} - 2$ for some A. Recalling (d), we find $p_1(t) = -2 + 2e^{2-t}$. The maximum condition requires that

$$\widehat{u}(t) \in \arg\max_{v\in[0,2]} \{(p_1(t) + 3p_2(t))\, v\} = \arg\max_{v\in[0,2]} \left\{(2e^{2-t} - 5)v\right\}.$$

The coefficient of v on the right is a decreasing function that changes sign when $2e^{2-t} = 5$, i.e., when $t = \tau \stackrel{\text{def}}{=} 2 - \ln(5/2)$. Hence the optimal control is given by

$$\widehat{u}(t) = \begin{cases} 2, & \text{if } 0 \le t < \tau, \\ 0, & \text{if } \tau < t \le 2. \end{cases}$$

The corresponding trajectories can now be obtained by integrating the dynamic equations. The first equation implies that the optimal arc x in the original problem coincides with

$$x_1(t) = \begin{cases} 7e^t - 2, & \text{if } 0 \le t \le 2 - \ln(5/2), \\ (7 - 5e^{-2})e^t, & \text{if } 2 - \ln(5/2) < t \le 2. \end{cases}$$

Although the problem statement does not require it, we may observe that

$$x_2(t) = \begin{cases} 10t - 14(e^t - 1), & \text{if } 0 \le t \le 2 - \ln(5/2), \\ 30 - 10\ln(5/2) - 2(7 - 5e^{-2})e^t, & \text{if } 2 - \ln(5/2) < t \le 2, \end{cases}$$

so the minimum value in the original problem is $x_2(2) = 40 - 10\ln(5/2) - 14e^2$.