M403(2012) Solutions—Problem Set 7

1. Since $x(2)$ is unconstrained, all extremals are normal. So here we consider

$$H(x,p,u) = p(x+u) - (2x - 3u - \frac{a}{2}u^2) = (p-2)x + (p+3)u + \frac{a}{2}u^2.$$

A control $\widehat{u}(\cdot)$ with response $x(\cdot)$ is extremal iff some $p(\cdot)$ obeys

(a) $-\dot{p}(t) = H_x(x(t),p(t),\widehat{u}(t)) = p(t) - 2$, so $\dot{p}(t) = -p(t) + 2$, giving

$$p(t) = Ke^{-t} + 2, \qquad K \in \mathbb{R}.$$

(b) $\dot{x}(t) = H_p(x(t),p(t),\widehat{u}(t)) = x(t) + \widehat{u}(t)$, as expected.

(c) $\widehat{u}(t) \in \arg\max_v \left\{(p(t)+3)v + \frac{a}{2}v^2 \;:\; 0 \le v \le 2\right\}$.

(d) $-p(2) = \nabla \ell_0(x(2)) = 0$, since here we have $\ell_0(x) \equiv 0$.

Conditions (a) and (c) imply $0 = Ke^{-2} + 2$, i.e., $K = -2e^2$, so $p(t) = 2 - 2e^{2-t}$.

Case $a = 0$ Here the maximum condition in (c) contains no quadratic terms, so the maximizer is determined by the sign of the coefficient $p(t) + 3 = 5 - 2e^{2-t}$. This quantity is increasing, with final value $p(2) + 3 = 3$ and initial value $p(0) + 3 = 5 - 2e^2 < 0$. Its unique zero occurs at the point $\theta \in (0,2)$ defined by

$$0 = 5 - 2e^{2-\theta}, \quad \text{i.e.,} \quad e^{-\theta} = \frac{5}{2}e^{-2}, \quad \text{i.e.,} \quad \theta = 2 - \ln\left(\frac{5}{2}\right).$$

$$\text{Thus} \qquad \widehat{u}(t) = \begin{cases} 0, & \text{if } p(t)+3 < 0, \text{ i.e., } 0 \le t < \theta, \\ \text{arb}, & \text{if } p(t)+3 = 0, \text{ i.e., } t = \theta, \\ 2, & \text{if } p(t)+3 > 0, \text{ i.e., } \theta < t \le 2. \end{cases}$$

The arbitrariness of $\widehat{u}$ at a single instant makes no difference to the state evolution, which clearly obeys $x(t) = 5e^t$ for $0 \le t < \theta$. On the final interval, where $\dot{x} = x + 2$, the general solution has the form $x(t) = Me^t - 2$, and continuity at $t = \theta$ gives

$$5e^\theta = Me^\theta - 2, \quad \text{i.e.,} \quad M = 5 + 2e^{-\theta} = 5 + 5e^{-2}.$$

So the unique extremal evolution (and a true minimum, by convexity) is

$$x(t) = \begin{cases} 5e^t, & \text{for } 0 \le t \le \theta, \\ 5e^t + 5e^{t-2} - 2, & \text{for } \theta < t \le 2. \end{cases}$$

Case $a = 1$ When the maximum condition in (c) contains the quadratic term with $a = 1$, it involves a *convex* function of v. Any critical point of such a function will give a *minimum,* not a maximum, so the maximizer will always lie at an endpoint of the control set $[0,2]$. Thus we find

$$\widehat{u}(t) = 2 \quad \text{when} \quad (p(t)+3)(2) + \tfrac{1}{2}(2)^2 > (p(t)+3)(0) + \tfrac{1}{2}(0)^2, \quad \text{i.e.,} \quad p(t) > -4,$$
$$\widehat{u}(t) = 0 \quad \text{when} \quad (p(t)+3)(0) + \tfrac{1}{2}(0)^2 > (p(t)+3)(2) + \tfrac{1}{2}(2)^2, \quad \text{i.e.,} \quad p(t) < -4.$$

We recall that $p(t) = 2(1-e^{2-t})$ is an increasing function with $p(0) = 2(1-e^2) < -4$ and $p(2) = 0$, so again the control has the form

$$\widehat{u}(t) = \begin{cases} 0, & \text{if } 0 \le t < \tau, \\ \text{arb}, & \text{if } t = \tau, \\ 2, & \text{if } \tau < t \le 2. \end{cases}$$

In this case, τ is defined by

$$-4 = p(\tau) = 2 - 2e^{2-\tau}, \quad \text{i.e.,} \quad e^{2-\tau} = 3 \quad \text{i.e.,} \quad \tau = 2 - \ln(3).$$

The calculation of $x(\cdot)$ is very similar to the one above. It yields

$$x(t) = \begin{cases} 5e^t, & \text{for } 0 \le t \le \tau, \\ 5e^t + 6e^{t-2} - 2, & \text{for } \tau < t \le 2. \end{cases}$$

2. Here $\quad H(t,x,p,u) := pxu - e^{-\delta t}(ux - x) = (p - e^{-\delta t})xu + xe^{-\delta t}, \quad$ so a control $\widehat{u}$ and associated response x are extremal if and only if some adjoint arc p obeys

(a) $-\dot{p}(t) = H_x(t, x(t), p(t), \widehat{u}(t)) = p(t)\widehat{u}(t) - e^{-\delta t}(\widehat{u}(t) - 1)$,

(c) $\widehat{u}(t) = \begin{cases} 0, & \text{if } \sigma(t) < 0, \\ ?, & \text{if } \sigma(t) = 0, \\ 1, & \text{if } \sigma(t) > 0, \end{cases} \qquad$ where $\quad \sigma(t) = p(t) - e^{-\delta t}$,

(d) $p(\pi) = 0$.

Here we have simplified the switching function σ defined in (c) by observing that $x(t) > 0$ for all t: this follows from the dynamics and the given inequality $\xi > 0$, since

$$x(t) = \xi \exp\left(\int_0^t \widehat{u}(r)\, dr \right).$$

From (d), we have $\sigma(\pi) = -e^{-\delta\pi} < 0$, so on some final interval of the form $(\theta, \pi]$ we have $\sigma(t) < 0$ and $\widehat{u}(t) = 0$. The corresponding state is constant at $x(\theta)$; we find the costate using (a) and (d):

$$-\dot{p}(t) = e^{-\delta t} \implies p(t) = \frac{1}{\delta}e^{-\delta t} + K \implies p(t) = \frac{1}{\delta}\left[e^{-\delta t} - e^{-\delta\pi}\right], \qquad t \in (\theta, \pi].$$

With this choice, we have

$$\sigma(t) = p(t) - e^{-\delta t} = \left(\frac{1}{\delta} - 1\right)e^{-\delta t} - \frac{1}{\delta}e^{-\delta\pi} = \left(\frac{1-\delta}{\delta}\right)\left[e^{-\delta t} - \frac{e^{-\delta\pi}}{1-\delta}\right], \qquad t \in (\theta, \pi].$$

This function is decreasing, and takes the value 0 when t obeys

$$e^{-\delta t} = \frac{e^{-\delta\pi}}{1-\delta}, \quad \text{i.e.,} \quad -\delta t = -\delta\pi - \ln(1-\delta) \quad \text{i.e.,} \quad t = \pi + \frac{1}{\delta}\ln(1-\delta).$$

Now $\delta^{-1}\ln(1-\delta) \to -\infty$ as $\delta \to 1^-$, so for large $\delta \in (0,1)$ we have $\theta = 0$ and control $\widehat{u} \equiv 0$ is extremal. But $\delta^{-1}\ln(1-\delta) \to -1$ as $\delta \to 0^+$, so for small $\delta \in (0,1)$ we have $\theta \approx \pi - 1$; in detail,

$$\theta = \pi + \frac{1}{\delta}\ln(1-\delta).$$

The qualitative change occurs when $\theta = 0$, i.e., at the δ-value in $(0, 1)$ satisfying

$$\delta\pi = -\ln(1-\delta). \qquad \text{[Solution: } \delta \approx 0.94933.]$$

Notice that $\sigma(t) = p(t) - e^{-\delta t}$ and (a) together imply

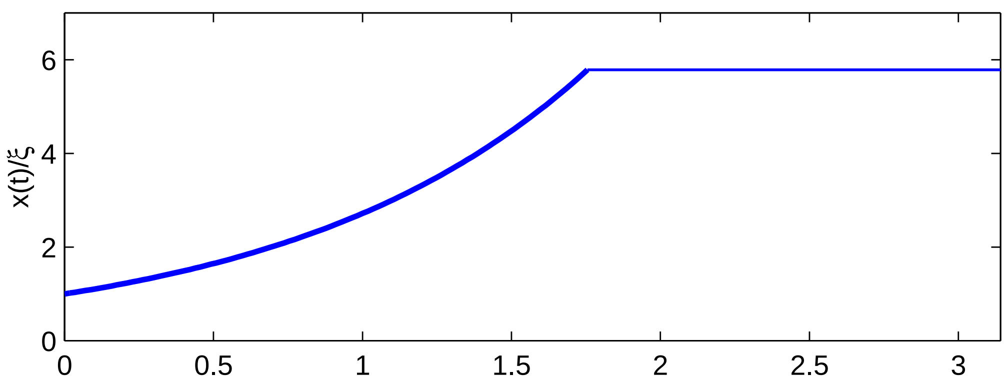
$$\dot\sigma(t) = \dot p(t) + \delta e^{-\delta t} = -p\widehat{u} + e^{-\delta t}(\widehat{u} - 1) - \delta e^{-\delta t} = -\widehat{u}\sigma - (1+\delta)e^{-\delta t}.$$

This shows that at points near $t = \theta$ where $\widehat{u}$ is continuous, $\dot\sigma(t) < 0$. Hence σ has a simple sign-change at $t = \theta$, and obeys $\sigma(t) > 0$ for $t \in (0, \theta)$. It follows that $\widehat{u}(t) = 1$ in this interval. The unique extremal control is now completely known:

$$\widehat{u}(t) = \begin{cases} 1, & \text{if } 0 \le t < \theta, \\ 0, & \text{if } \theta < t \le 1, \end{cases} \qquad \theta = \max\left\{0, \pi + \delta^{-1}\ln(1-\delta)\right\}.$$

When $\delta = \frac{1}{2}$, $\theta = \pi - 2\ln 2$ and

$$x(t) = \begin{cases} \xi e^t, & \text{for } 0 \le t \le \theta, \\ \xi e^{\pi}/4, & \text{for } \theta \le t \le \pi. \end{cases}$$



3. Here $\ell_0(x) = 2x$, $L_0(t, x, u) = x^2 - u^2$, and the preHamiltonian is

$$H(t, x, p, u) = pxu - x^2 - u^2.$$

The extremality conditions for a control $\widehat{u}(\cdot)$ with corresponding state $x(\cdot)$ apply in normal form ($\lambda_0 = 1$) because there are no constraints on the right endpoint. They say that some arc $p(\cdot)$ obeys

(i) $\dot{\widehat{h}}(t) = H_t(t, x(t), p(t), \widehat{u}(t)) = 0, \quad \text{where} \quad \widehat{h}(t) = H(t, x(t), p(t), \widehat{u}(t)),$
$-\dot p(t) = H_x(t, x(t), p(t), \widehat{u}(t)) = p(t)\widehat{u}(t) - 2x(t),$

(ii) $\dot x(t) = H_p(t, x(t), p(t), \widehat{u}(t)) = x(t)\widehat{u}(t), \qquad x(0) = \xi,$

(iii) $\widehat{u}(t) \in \arg\max\left\{p(t)x(t)v - v^2 \,:\, v \in \mathbb{R}\right\},$

(iv) $-p(T) = \nabla\ell_0(x(T)) = 2.$

From (iii), we get $2\widehat{u}(t) = p(t)x(t)$. This reduces (i)(ii)(iv) to the following system:

$$\begin{aligned} \dot x(t) &= \tfrac{1}{2}p(t)x(t)^2, & x(0) &= \xi, \\ \dot p(t) &= \tfrac{1}{2}x(t)\left(4 - p(t)^2\right), & p(T) &= -2. \end{aligned} \qquad (*)$$

Notice that $\dot{p}(t) = 0$ whenever $p(t)^2 = 4$, and this situation arises at time T. Thus (by "coincidence") we obtain a solution to $(*)$ with *constant* costate $p(t) = -2$ if and only if

$$\dot{x}(t) = -x(t)^2, \qquad x(0) = \xi$$

This is separable; integration over $[0, t]$ gives

$$-\int_{x(0)}^{x(t)} \frac{dx}{x^2} = \int_{t'=0}^{t} dt' \implies \frac{1}{x(t)} - \frac{1}{\xi} = t \implies x(t) = \frac{1}{t + (1/\xi)} = \frac{\xi}{1 + \xi t}.$$

This is an extremal state on any interval $[0, T]$, $T > 0$; in particular, when $T = 2$.

(a) When $\xi = 1$ an extremal control and state are given by

$$\widehat{u}(t) = \frac{\dot{x}(t)}{x(t)} = -\frac{1}{1+t}; \qquad x(t) = \frac{1}{1+t}.$$

(b) When $\xi = -1$ the calculations above give $x(t) = -(1-t)^{-1}$. This is a legitimate extremal on $[0, T]$ when $T < 1$, but it blows up at $t = 1$, so it is not an extremal on $[0, T]$ for any $T \geq 1$.

4. (i) If $\alpha + \beta = 0$, then $\alpha = -\beta \neq 0$ and the dynamics reduce to $\dot{x} = (\alpha - \alpha u)\, x = -\alpha x\, (u - 1)$. Hence we have the same objective value for every admissible control, namely,

$$\int_0^1 x(t)\, [u(t) - 1]\; dt = \int_0^1 -\frac{1}{\alpha} \dot{x}(t)\, dt = -\frac{1}{\alpha} \left(\Xi - \xi\right).$$

Thus every admissible control is optimal. This case is not too interesting, so we restrict attention to the case $\alpha + \beta \neq 0$ in what follows.

(ii) Here the pre-Hamiltonian is

$$H(x, p, u) = p(\alpha + \beta u)x - \lambda_0 x(u-1) = (\alpha p + \lambda_0)x + (\beta p - \lambda_0)xu.$$

An extremal control u must come with some arc p and scalar $\lambda_0 \in \{0, 1\}$, not both zero, such that

(a) $-\dot{p}(t) = H_x(x(t), p(t), u(t)) = \alpha p(t) + \lambda_0 + (\beta p(t) - \lambda_0) u(t)$,

(c) $u(t) \in \arg\max_{v \in [0,1]} \left\{ (\beta p(t) - \lambda_0)\, x(t) v \right\}$.

If we define $\sigma(t) = (\beta p(t) - \lambda_0) x(t)$, then (ii) implies

$$u(t) \in \begin{cases} \{1\}, & \text{if } \sigma(t) > 0, \\ [0, 1], & \text{if } \sigma(t) = 0, \\ \{0\}, & \text{if } \sigma(t) < 0. \end{cases}$$

To show that σ is monotonic, we compute

$$\begin{aligned} \dot{\sigma}(t) &= \beta \dot{p}(t) x(t) + (\beta p(t) - \lambda_0)\, \dot{x}(t) \\ &= -\beta\, [\alpha p + \lambda_0 + (\beta p - \lambda_0) u]\, x + (\beta p - \lambda_0)\, [\alpha + \beta u]\, x \\ &= [\, -\alpha\beta p - \lambda_0 \beta - \beta(\beta p - \lambda_0) u + \alpha\beta p - \lambda_0 \alpha + \beta(\beta p - \lambda_0) u]\, x \\ &= -\lambda_0\, (\alpha + \beta)\, x(t). \end{aligned}$$

Observe that $x(t) = \xi \exp\left(\int_0^t [\alpha + \beta u(r)]\, dr \right)$ never changes sign, so $\dot{\sigma}$ never changes sign, so σ is monotonic.

(iii) Suppose $\lambda_0 = 0$. Then $\dot{\sigma} \equiv 0$, so that σ is constant. If $\sigma > 0$, then $u \equiv +1$ gives $x(t) = \xi e^{(\alpha+\beta)t}$ and forces $\Xi = \xi e^{\alpha+\beta}$. This contradicts our choice of Ξ. Similarly, if $\sigma < 0$, then $u(t) \equiv 0$ and $x(t) = \xi e^{\alpha t}$. This forces $\Xi = \xi e^{\alpha}$. Again our choice of Ξ is violated. Finally, if $\sigma = 0$, then we must have $\beta p(t)x(t) \equiv 0$. Since $\beta \neq 0$ by assumption and $x(t) \neq 0$ for all t, it follows that $p \equiv 0$. This contradicts the phrase "not both zero" in the definition of extremality.

The case $\lambda_0 = 0$ is impossible, so we must have $\lambda_0 = 1$. This implies that $\dot{\sigma}(t) = -(\alpha+\beta)x(t)$ is nonzero and of constant sign, so σ is strictly monotonic.

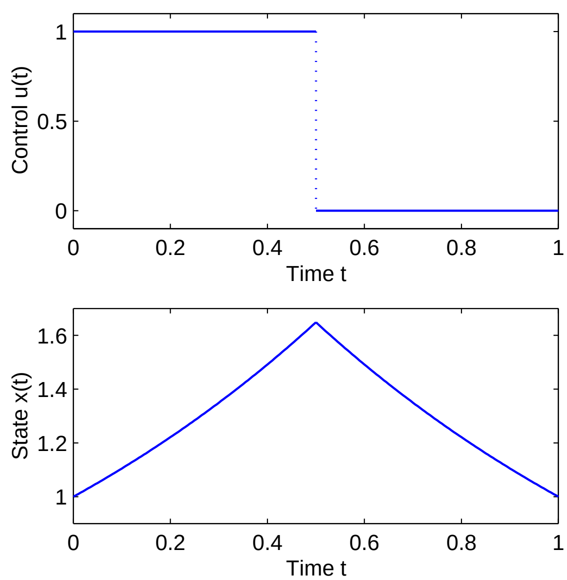
(iv) The analysis in (iii) shows that the constant controls $u \equiv 0$ and $u \equiv 1$ are not admissible, so the optimal control must switch. Since the sign of $\dot{\sigma}$ is the same as the sign of $-\xi(\alpha+\beta)$, the switching direction is easy to predict.

If $\xi(\alpha+\beta) > 0$, then $\dot{\sigma}(t) < 0$, so for some $\tau \in (0,1)$,

$$u(t) = \begin{cases} 1, & 0 \le t < \tau, \\ 0, & \tau < t \le 1, \end{cases} \qquad \text{giving} \qquad x(t) = \begin{cases} \xi e^{(\alpha+\beta)t}, & 0 \le t \le \tau, \\ \xi e^{(\alpha+\beta)\tau} e^{\alpha(t-\tau)}, & \tau < t \le 1. \end{cases}$$

To find τ we apply the endpoint constraint:

$$\Xi = \xi e^{\beta\tau+\alpha} \implies \tau = \frac{1}{\beta}\left(\ln\left(\frac{\Xi}{\xi}\right) - \alpha\right).$$



Question 4: Extremal Control and State for $\alpha = -1$, $\beta = 2$, $\xi = 1$, $\Xi = 1$.

If $\xi(\alpha+\beta) < 0$, then $\dot{\sigma}(t) > 0$, so for some $\tau \in (0,1)$,

$$u(t) = \begin{cases} 0, & 0 \le t < \tau, \\ 1, & \tau < t \le 1, \end{cases} \qquad \text{giving} \qquad x(t) = \begin{cases} \xi e^{\alpha t}, & 0 \le t \le \tau, \\ \xi e^{\alpha\tau} e^{(\alpha+\beta)(t-\tau)}, & \tau < t \le 1. \end{cases}$$

Again, it is easy to solve for τ:

$$\Xi = \xi e^{\alpha+\beta-\beta\tau} \implies \tau = \frac{1}{\beta}\left((\alpha+\beta) - \ln\left(\frac{\Xi}{\xi}\right)\right).$$

Specific Case $\alpha = -1$, $\beta = 2$, $\xi = 1$, $\Xi = 1$: Here $\xi(\alpha+\beta) > 0$, so $\tau = \frac{1}{2}\left(\ln(1) - (-1)\right) = \frac{1}{2}$. A sketch of the unique extremal control and corresponding state is provided.

5. Here $H(x,y,p,q,u) = p\cos u + q\sin u - \lambda_0 x\sin u$. For an extremal control $\widehat{u}$ with associated response x, there exist a constant $\lambda_0 \in \{0,1\}$ and an arc p on $[0,T]$, not both zero, obeying

(a) the adjoint equations

$$-\dot{p}(s) = \widehat{H}_x(s) = -\lambda_0 \sin\widehat{u}(s),$$
$$-\dot{q}(s) = \widehat{H}_y(s) = 0,$$

(b) the state equations written in the problem statement,

(c) the maximum condition, i.e., $\widehat{u}(s)$ maximizes $(p(s), q - \lambda_0 x(s)) \bullet (\cos v, \sin v)$ over all real v, so

$$(\cos\widehat{u}(s), \sin\widehat{u}(s)) = \frac{(p(s), q - \lambda_0 x(s))}{|(p(s), q - \lambda_0 x(s))|}, \quad \text{and}$$

(d) the transversality condition, which provides no information.

Notice that condition (a) implies that the adjoint function $q(s)$ is actually constant.

(i) If $\lambda_0 = 0$ in conditions (a)–(d) above, then condition (a) implies that the adjoint function $p(s)$ is constant. It follows from (c) that the steering direction $(\dot{x}(s), \dot{y}(s)) = (\cos\widehat{u}(s), \sin\widehat{u}(s))$ is constant too. To satisfy the endpoint constraints, this constant must be $(1,0)$, and the constraints must be set up so that the evolution produced by this input hits the target. Thus an abnormal extremal is possible if and only if $T = B$, and in this case it is given by $\widehat{u} \equiv 0$, $(x(s), y(s)) = (s, 0)$.

(ii) If $\lambda_0 = 1$, then the adjoint equation (a) implies $\dot{p}(s) = \sin\widehat{u}(s) = y(s)$, so that $p(s) = y(s) - a$ for some constant a. Consequently

$$\begin{aligned}
(\dot{x}(s), \dot{y}(s)) &= (\cos\widehat{u}(s), \sin\widehat{u}(s)) && \text{by the dynamics,}\\
&= \frac{(p(s), q - x(s))}{|(p(s), q - x(s))|} && \text{by the maximum condition (c)}\\
&= \frac{(y(s) - a, q - x(s))}{|(y(s) - a, q - x(s))|} && \text{by the observation above.}
\end{aligned}$$

This shows that the vector $(\dot{x}(s), \dot{y}(s))$ is parallel to $(y(s) - a, q - x(s))$, and hence perpendicular to $(x(s) - q, y(s) - a)$, which implies

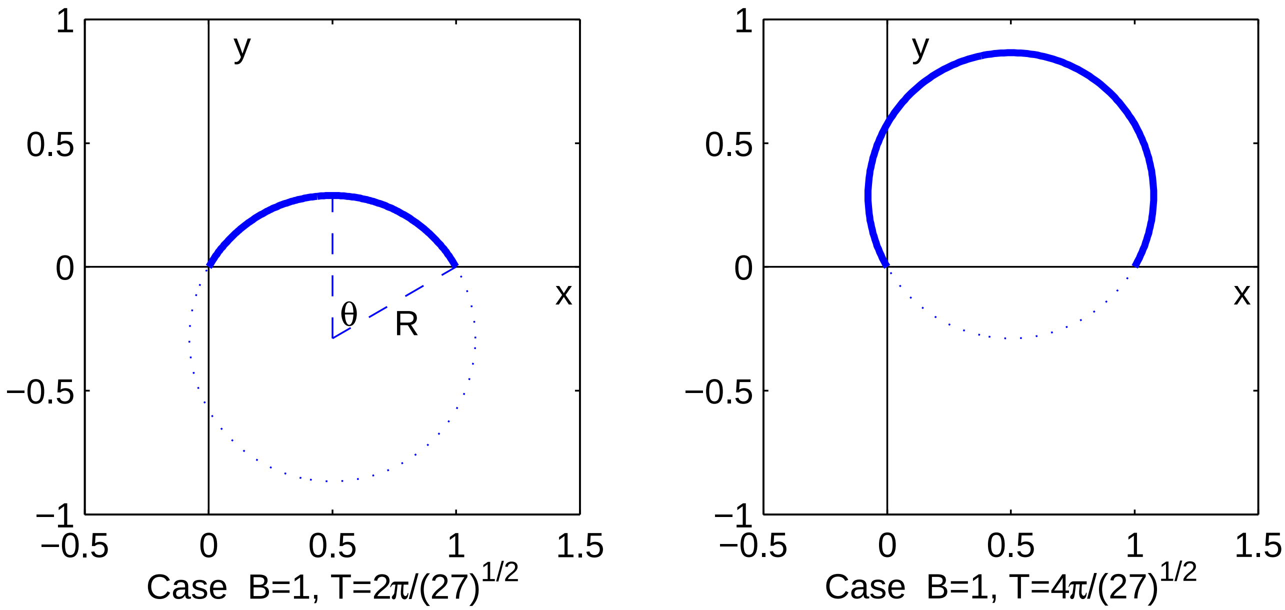
$$0 = (\dot{x}(s), \dot{y}(s)) \bullet (x(s) - q, y(s) - a) = \frac{d}{ds}\left[\tfrac{1}{2}(x(s) - q)^2 + \tfrac{1}{2}(y(s) - a)^2\right].$$

In other words, there is some constant $R > 0$ such that

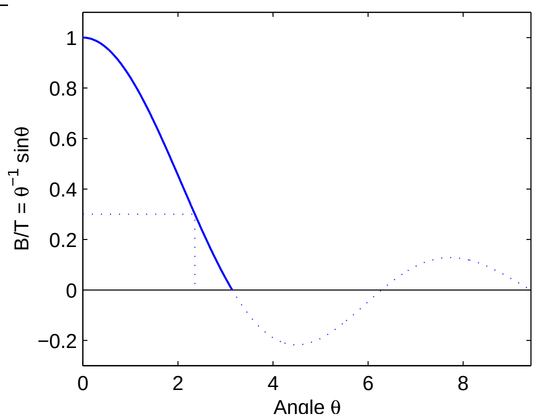
$$(x(s) - q)^2 + (y(s) - a)^2 = R^2.$$

This shows that the extremal trajectory forms an arc of a circle in the (x,y)-plane. To identify the circle, we notice that $\dot{x}(s)^2 + \dot{y}(s)^2 = 1$, so that the positive constant T can be interpreted as the total arc length of the extremal curve. Together with the endpoint conditions $(x(0), y(0)) = (0,0)$ and $(x(T), y(T)) = (B, 0)$ this implies that the desired circle has its centre at the point $(x,y) = (B/2, -R\cos\theta)$, where the radius R and half- angle θ are related by the arc length constraint $R\theta = T$ and the trigonometric constraint $R\sin\theta = B/2$. Rearranging these equations leads to a nonlinear relation that determines θ in terms of the ratio B/T, and a second equation that determines R once θ is known:

$$\frac{\sin\theta}{\theta} = \frac{B}{T}, \qquad R = \frac{T}{2\theta}.$$



Question 5: Extremal arcs for $B = 1$ and different T-values.



To find θ, solve $\sin(\theta)/\theta = B/T$.

Sketches of two possible extremals are shown below, together with a graph of $\theta^{-1}\sin\theta$ that can be inverted to determine θ as mentioned above.

(iii) Removing the right endpoint constraint on $x(T)$ produces a useful piece of information from the transversality condition, namely, $p(T) = 0$. In the normal case, we already know that $p(s) = y(s) - a$ always, so the constraint $y(T) = 0$ then implies $a = 0$. The resulting circular arc has its centre on the x-axis, at the point $(x, y) = (B/2, 0)$. To satisfy the endpoint constraints, our extremal must be the arc of a semicircle. The optimal endpoint is $x(T) = 2T/\pi$.

Abnormal extremals are impossible in this case because taking $\lambda_0 = 0$ together with the endpoint condition $p(T) = 0$ gives $p \equiv 0$, from which the maximum condition implies that $\dot{y}(s) = \sin \widehat{u}(s)$ is constant at either -1 or $+1$. Either way, it is impossible to satisfy the endpoint constraint $y(T) = 0$.