Math 403 Problem Set 8

Due in class on Wednesday 21 November 2012

1. The problem of stopping a controlled harmonic oscillator in prescribed time T with minimum energy can be expressed succinctly as follows:

$$\begin{aligned} \text{minimize} \quad & J[u] \stackrel{\text{def}}{=} \int_0^T \tfrac{1}{2}|u(t)|^2\,dt, \\ \text{over all} \quad & u\colon [0,T] \to \mathbb{R} \text{ piecewise continuous} \\ \text{subject to} \quad & \ddot{y}(t) + y(t) = u(t), \ \text{a.e. } t \in [0,T], \\ & y(0) = y_0,\ \dot{y}(0) = v_0, \\ & y(T) = 0,\ \dot{y}(T) = 0. \end{aligned}$$

Apply the Maximum Principle to identify an extremal control.

Preview: You will find a linear combination of sinusoids for $\widehat{u}$. The coefficients in this combination will be determined by the problem data y_0, v_0, and T. The explicit form of the coefficients is a little nasty, so don't work it out. Just present a 2×2 system of linear equations they must satisfy and assume the reader can solve it.

2. All three parts below involve these familiar dynamics:

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = u(t); \qquad x_1(0) = 0,\ x_2(0) = 0.$$

(a) Choose $T > 0$ and $u\colon [0,T] \to \mathbb{R}$ to arrange $x_1(T) = 69$, $x_2(T) = 0$ while minimizing the quantity

$$J(T, u(\cdot)) \stackrel{\text{def}}{=} \int_0^T \left(1 + \tfrac{1}{2}u(t)^2\right)\,dt.$$

(b) Repeat part (a), but impose the additional constraint $|u(t)| \le 1$ for almost every $t \in [0,T]$.

(c) Choose $T > 0$ and $u\colon [0,T] \to [-1,1]$ to arrange $x_1(T) = 69$, $x_2(T) = 0$ while minimizing the quantity

$$G(T, u(\cdot)) \stackrel{\text{def}}{=} \int_0^T (1 + |u(t)|)\,dt.$$

(This problem differs from (b) only in the change from $\frac{1}{2}u^2$ to $|u|$ in the performance criterion. Note that $|u(t)| \le 1$ a.e. is required.)

3. Discuss extremal controls for the constrained calculus-of-variations problem with planning duration $T = 10$ and initial state $\xi = 1$, given the general form

$$\begin{aligned} \text{minimize} \quad & \ell(x(T)) + \int_0^T e^{-\delta t}\left[M(x(t)) + N(x(t))\dot{x}(t)\right]\,dt \\ \text{subject to} \quad & A(x(t)) \le \dot{x}(t) \le B(x(t)) \quad \text{a.e. } t \in [0,T], \\ & x(0) = \xi. \end{aligned}$$

Use the ingredients below, assuming k and δ are constants:

$$A(x) = -x, \quad B(x) = 1, \quad \ell(x) = kx, \quad M(x) = -2\delta x, \quad N(x) = x.$$

Make all the qualitative statements you can in cases where $\delta > 0$ is not too large and k may have either sign. Then provide a completely explicit solution (perhaps with computer assistance) for the case where $\delta = 1/10$ and $k = 1/2$.