M403(2012) Solutions—Problem Set 8

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1. The given definitions, namely, $x_1 = y,\ x_2 = \dot{y},\ x_3 = \frac{1}{2}\int_0^t u(r)^2\,dr,$ lead to the following alternative formulation of this problem:

$$\begin{aligned} \text{minimize} \quad & \ell(x_1(T), x_2(T), x_3(T)), \qquad \text{where } \ell(x,y,z) = z, \\ \text{over all} \quad & u\colon [0,T] \to \mathbb{R} \text{ piecewise continuous} \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \qquad x_1(0) = x_0, \\ & \dot{x}_2(t) = -x_1(t) + u(t), \qquad x_2(0) = v_0, \\ & \dot{x}_3(t) = \tfrac{1}{2}u(t)^2, \qquad x_3(0) = 0, \\ & (x_1(T), x_2(T), x_3(T)) \in S \stackrel{\text{def}}{=} \{(0,0,r) \,:\, r \in \mathbb{R}\}. \end{aligned}$$

We analyse the extremality conditions for a control $\widehat{u}(\cdot)$ with response trajectory $\mathbf{x}(\cdot)$. These involve the preHamiltonian

$$H(t, \mathbf{x}, \mathbf{p}, u) = p_1 x_2 - p_2 x_1 + p_2 u + \tfrac{1}{2} p_3 u^2.$$

(i) The costate equations expand into a 3×3 system:

$$-\dot{p}_1 = H_{x_1} = -p_2, \qquad -\dot{p}_2 = H_{x_2} = p_1, \qquad -\dot{p}_3 = H_{x_3} = 0.$$

The third equation gives $p_3(t) = -\mu$ for some constant μ, while the first two combine to give constants A, B such that

$$\begin{aligned} \ddot{p}_1 = \dot{p}_2 = -p_1 \implies & p_1(t) = A\cos(t) + B\sin(t); \\ & p_2(t) = \dot{p}_1(t) = -A\sin(t) + B\cos(t). \end{aligned}$$

(ii) The state equations come second; we'll analyze them later.

(iii) The maximum condition says that for almost every t, $\widehat{u}(t)$ maximizes $H(t, \mathbf{x}(t), \mathbf{p}(t), v)$ over all $v \in \mathbb{R}$. Equivalently,

$$\widehat{u}(t) \in \arg\max \left\{ p_2(t) v - \frac{\mu}{2} v^2 \,:\, v \in \mathbb{R} \right\}.$$

(iv) To decode the transversality condition $-\mathbf{p}(T) \in \nabla \ell(\mathbf{x}(T)) + N_S(\mathbf{x}(T))$, it is useful to observe that

$$\nabla \ell(x,y,z) \equiv (0,0,1), \qquad \text{and} \qquad \forall z \in S,\ N_S(z) = \{(\alpha, \beta, 0) \,:\, \alpha \in \mathbb{R},\ \beta \in \mathbb{R}\}.$$

Thus the transversality condition says that for some $\alpha, \beta \in \mathbb{R}$,

$$-p_1(T) = \alpha, \quad -p_2(T) = \beta, \quad -p_3(T) = 1 + 0.$$

The first two conditions give no information at all, since α, β are arbitrary, but the last one is useful: it gives $\mu = 1$.

Using $\mu = 1$ in (iii) leads to

$$0 = \frac{d}{dv}\left[p_2(t) v - \tfrac{1}{2} v^2 \right]_{v = \widehat{u}(t)} = p_2(t) - \widehat{u}(t), \quad \text{i.e.,} \quad \widehat{u}(t) = p_2(t) = -A\sin(t) + B\cos(t).$$

It's convenient to use the original variables to analyse the state response,

$$\ddot{y} + y = \widehat{u} = -A\sin(t) + B\cos(t).$$

A particular solution of this nonhomogeneous equation is

$$Y(t) = \frac{A}{2}t\cos(t) + \frac{B}{2}t\sin(t),$$

so the general solution is

$$y(t) = \alpha\cos(t) + \beta\sin(t) + Y(t), \qquad \alpha, \beta \in \mathbb{R}.$$

There are four constants to find here, and we have four conditions to use. The initial conditions specify

$$\begin{aligned} x_0 = y(0) = \alpha, &\qquad \text{so} \alpha = x_0 \\ v_0 = \dot{y}(0) = \beta + \tfrac{1}{2}A, &\text{so} \beta = v_0 - \tfrac{1}{2}A. \end{aligned}$$

Using these facts to simplify the final conditions, we get

$$\begin{aligned} 0 = y(T) &= x_0\cos(T) + (v_0 - \tfrac{1}{2}A)\sin(T) + Y(T) \\ &= x_0\cos(T) + v_0\sin(T) + \tfrac{1}{2}\left[T\cos(T) - \sin(T)\right]A + \tfrac{1}{2}T\sin(T)B \\ 0 = \dot{y}(T) &= -x_0\sin(T) + v_0\cos(T) - \tfrac{1}{2}T\sin(T)A + \left[\tfrac{1}{2}T\cos(T) + \sin(T)\right]B. \end{aligned}$$

This is a 2×2 system to solve for A and B:

$$\begin{aligned} \left[\sin(T) - T\cos(T)\right]A - T\sin(T)B &= 2x_0\cos(T) + 2v_0\sin(T) \\ T\sin(T)A - \left[T\cos(T) + \sin(T)\right]B &= 2v_0\cos(T) - 2x_0\sin(T). \end{aligned}$$

The coefficient matrix has determinant $T^2 - \sin^2(T)$, which is positive for all $T > 0$, so there is certain to be a unique solution for the pair (A, B). These constants uniquely determine the control function $\widehat{u}$.

2. These solutions treat the general constraint $x_1(T) = D > 0$ and specialize to $D = 69$ only after most of the work is done.

Case (a)—Quadratic cost, no control constraints: Here

$$H(t, x, p, u) = p_1x_2 + p_2u - \lambda_0\left(1 + \tfrac{1}{2}u^2\right).$$

A pair $(u(\cdot), x(\cdot))$ is extremal iff some $\lambda_0 \in \{0, 1\}$ and $p(\cdot)$, not both zero, obey

(i) $(\dot{h}(t), -\dot{p}_1(t), -\dot{p}_2(t)) = (H_t(\cdots), H_{x_1}(\cdots), H_{x_2}(\cdots)) = (0, 0, p_1(t))$.

This gives $h(t) \equiv h$, $p_1(t) \equiv -m$ and $p_2(t) = mt + b$ for some constants h, m, b.

(iii) $u(t) \in \arg\max_{v\in\mathbb{R}} H(t, x(t), p(t), v) = \arg\max_{v\in\mathbb{R}} \left\{p_2(t)v - \tfrac{1}{2}\lambda_0 v^2\right\}$.

If $\lambda_0 = 0$, this forces $p_2(t) \equiv 0$, which implies $p_1 = 0$. Thus both λ_0 and $p(\cdot)$ are zero, a case explicitly ruled out by the Maximum Principle. So we may assume $\lambda_0 = 1$, and report the maximizing control-value as

$$u(t) = p_2(t) = mt + b.$$

(ii) $(\dot{x}_1(t), \dot{x}_2(t)) = (x_2(t), u(t))$.

Using the initial conditions $x_1(0) = 0 = x_2(0)$, we integrate to get

$$\begin{aligned}\dot{x}_2(t) &= mt + b \implies x_2(t) = \tfrac{1}{2}mt^2 + bt,\\ \dot{x}_1(t) &= x_2(t) \quad\implies x_1(t) = \frac{m}{6}t^3 + \frac{b}{2}t^2.\end{aligned}$$

(iv) $(h(T), -p_1(T), -p_2(T)) = \nabla\widetilde{\ell}(T, x_1(T), x_2(T)) = (0, \lambda_1, \lambda_2)$.

This comes from the endpoint cost and constraints, captured by defining

$$\ell_0(t, x_1, x_2) = 0, \qquad \ell_1(t, x_1, x_2) = x_1 - 69, \qquad \ell_2(t, x_1, x_2) = x_2.$$

The only useful piece of information here is $h = 0$.

Along the extremal trajectory, the explicit forms for $x(t)$, $p(t)$, and $u(t)$ above give

$$\begin{aligned}h(t) = H(t, x(t), p(t)) &= p_1(t)x_2(t) + p_2(t)u(t) - 1 - \tfrac{1}{2}u(t)^2\\ &= -m\left(\tfrac{1}{2}mt^2 + bt\right) + (mt + b)^2 - 1 - \tfrac{1}{2}(mt + b)^2\\ &= \tfrac{1}{2}b^2 - 1.\end{aligned}$$

Knowing $h(t) \equiv 0$ gives $b^2 = 2$. Now the final-state conditions require

$$\begin{aligned}0 = x_2(T) &= \tfrac{1}{2}mT^2 + bT = \tfrac{1}{2}T(mT + 2b),\\ D = x_1(T) &= \frac{m}{6}T^3 + \frac{b}{2}T^2 = \frac{T^2}{6}\left[mT + 3b\right].\end{aligned}$$

Clearly the choice $m = 0$ does not produce a solution, so we have $mT = -2b$ from the first line, which reduces the second line to

$$D = \frac{4b^2/m^2}{6}\left[-2b + 3b\right] = \frac{2b^3}{3m^2}.$$

Since $D > 0$, we must have $b > 0$, so $b = \sqrt{2}$ and $m^2 = 2^{5/2}/(3D)$. A positive solution for T calls for $m < 0$, so we have

$$m = -\frac{2^{5/4}}{\sqrt{3D}}, \qquad T = -\frac{2b}{m} = \frac{2^{3/2}}{2^{5/4}/\sqrt{3D}} = 2^{1/4}\sqrt{3D}.$$

In summary, the extremal control-state pair and final time are uniquely specified by

$$u(t) = mt + b, \qquad x_1(t) = \frac{m}{6}t^3 + \frac{b}{2}t^2, \qquad x_2(t) = \frac{m}{2}t^2 + bt, \qquad 0 \le t \le T,$$

with m, b, and T as above. In the special case $D = 69$, these values are

$$b = \sqrt{2} \approx 1.41421, \quad m = -\frac{2^{5/4}}{3\sqrt{23}} \approx -0.165311, \quad T = 2^{1/4}3\sqrt{23} \approx 17.1097.$$

Case (b)—Quadratic cost, constrained controls: The constraint $|u(t)| \le 1$ does not affect the preHamiltonian or the endpoint criteria. So just as in part (a),

$$h(t) \equiv h, \qquad p_1(t) \equiv -m, \qquad p_2(t) = mt + b$$

for some constants h, m, b, with $h = 0$ from the transversality condition. However, the control constraint does enter the maximum condition:

$$u(t) \in \arg\max_{|v|\le 1} \left\{ p_2(t)v - \tfrac{1}{2}\lambda_0 v^2 \right\}.$$

Abnormal Case: If $\lambda_0 = 0$, the maximum condition requires

$$u(t) \in \arg\max_{|v|\le 1} \{p_2(t)v\} = \operatorname{sgn}(p_2(t)).$$

This implies $p_2(t)u(t) = |p_2(t)| = |mt + b|$, and leads to

$$\begin{aligned} h(t) &= H(t, x(t), p(t), u(t)) = p_1(t)x_2(t) + p_2(t)u(t) \\ \iff \qquad 0 &= -mx_2(t) + |mt + b|. \end{aligned}$$

Taking the limit as $t \to 0^+$ gives $0 = |b|$, so $b = 0$. Subsequently sending $t \to T^-$ gives $0 = |mT|$, so $m = 0$. Hence $p_1(t) = 0 = p_2(t)$ for all t and the nontriviality condition describing an extremal is not satisfied. We can safely reject the possiblity that $\lambda_0 = 0$ and search only for normal extremals. Let $\lambda_0 = 1$.

Normal Case: When $\lambda_0 = 1$, we have

$$u(t) \in \arg\max_{-1\le v\le 1} \left\{ p_2(t)v - \tfrac{1}{2}v^2 \right\}.$$

The function of v being maximized here is increasing on the interval $(-\infty, p_2(t))$ and decreasing on $(p_2(t), +\infty)$, so it is maximized by choosing the point of $[-1, 1]$ nearest to $p_2(t)$. That is,

$$u(t) = \operatorname{sat}(p_2(t)) = \operatorname{sat}(mt + b), \qquad \text{where} \qquad \operatorname{sat}(r) = \begin{cases} -1, & \text{if } r \le -1, \\ r, & \text{if } -1 < r < 1, \\ +1, & \text{if } r \ge +1. \end{cases}$$

(The name of this auxiliary function reflects engineering terminology: it is a standard "saturation" nonlinearity.) Since the function sat is nondecreasing, we deduce that $u(\cdot)$ is nondecreasing when $m > 0$ and nonincreasing when $m < 0$. If $m \ge 0$, we have a problem: this makes $\dot{x}_2(\cdot)$ a nondecreasing function, and then the endpoint conditions $x_2(0) = 0 = x_2(T)$ force $x_2(t) \le 0$ for all t. We would then have $x_1(T) = \int_0^T x_2(t)\,dt \le 0$ when we need $x_1(T) = D > 0$. So it must be the case that $m < 0$ and the extremal control has the form

$$u(t) = \begin{cases} 1, & \text{for } 0 \le t \le r, \\ mt + b, & \text{for } r < t < s, \\ -1, & \text{for } s \le t \le T, \end{cases}$$

where the transition times $r, s \in [0, T]$ are determined by

$$\begin{aligned} 1 &= p_2(r) = mr + b, \quad \text{i.e.,} \quad r = \frac{b-1}{(-m)}, \\ -1 &= p_2(s) = ms + b, \quad \text{i.e.,} \quad s = \frac{b+1}{(-m)}. \end{aligned}$$

Integrating $\dot{x}_2 = u$ and $\dot{x}_1 = x_2$ and enforcing continuity for $x_1(\cdot)$ and $x_2(\cdot)$ leads to

$$\begin{aligned} 0 \le t \le r: &\qquad x_2(t) = t, \\ r \le t \le s: &\qquad x_2(t) = \frac{m}{2}\left(t^2 - r^2\right) + b(t - r) + r, \\ s \le t \le T: &\qquad x_2(t) = \frac{m}{2}\left(s^2 - r^2\right) + (1 + b)s + (1 - b)r - t. \end{aligned}$$

Now consider the function

$$h(t) = H(t, x(t), p(t), u(t)) = -mx_2(t) + (mt + b)u(t) - 1 - \tfrac{1}{2}u(t)^2.$$

Analyzing the condition $h(t) = 0$ on the first subinterval identified above gives

$$\forall t \in (0, r), \qquad 0 = h(t) = -m(t) + (mt + b)(1) - \frac{3}{2} = b - \frac{3}{2} \implies b = \frac{3}{2}.$$

Therefore

$$r = -\frac{1}{2m}, \qquad s = -\frac{5}{2m}.$$

Using these values at the right end gives

$$\begin{aligned} 0 &= 2mx_2(T) = (ms)^2 - (mr)^2 + 5ms - mr - 2mT \\ &= \frac{25}{4} - \frac{1}{4} - \frac{25}{2} + \frac{1}{2} - 2mT, \quad \text{i.e.,} \quad mT = -3. \end{aligned}$$

Back-substitution gives

$$r = -\frac{1}{2m} = \frac{T}{6}, \qquad s = -\frac{5}{2m} = \frac{5T}{6}.$$

The only constant left to find is T, which must be the time when $x_1(T) = D$. Using the constants we have already found leads to

$$\begin{aligned} 0 \le t \le \frac{T}{6} &: \qquad x_2(t) = t, \\ \frac{T}{6} \le t \le \frac{5T}{6} &: \qquad x_2(t) = -\frac{3}{2T}t^2 + \frac{3}{2}t - \frac{T}{24}, \\ \frac{5T}{6} \le t \le T &: \qquad x_2(t) = T - t. \end{aligned}$$

Therefore

$$\begin{aligned} 0 \le t \le \frac{T}{6} &: \qquad x_1(t) = \tfrac{1}{2}t^2, \\ \frac{T}{6} \le t \le \frac{5T}{6} &: \qquad x_1(t) = -\frac{1}{2T}t^3 + \frac{3}{4}t^2 - \frac{T}{24}t + \frac{T^2}{432}, \\ \frac{5T}{6} \le t \le T &: \qquad x_1(t) = \frac{5T^2}{72} + \frac{T}{3} - \tfrac{1}{2}(t - T)^2. \\ \frac{5T}{6} \le t \le T &: \qquad x_1(t) = Tt - \tfrac{1}{2}t^2 - \frac{31T^2}{108}. \end{aligned}$$

Arranging $x_1(T) = D$ requires $108x_1(T) = 108D$. We take the positive solution of

$$108D = 54T^2 - 31T^2 = 23T^2, \quad \text{i.e.,} \quad T = \sqrt{\frac{108D}{23}}.$$

In the special case $D = 69$, we have $T = \sqrt{\dfrac{108 \times 69}{23}} = \sqrt{324} = 18$.

(c) Now $H(t, x, p, u) = p_1x_2 + p_2u - \lambda_0(1 + |u|)$. For $(u(\cdot), x(\cdot))$ to be extremal on $[0, T]$, there must be some $\lambda_0 \in \{0, 1\}$ and arc $p(\cdot)$, not both zero, such that

$$p_1(t) = -m, \quad p_2(t) = mt + b, \quad 0 = H(t, x(t), p(t), u(t)), \qquad \text{a.e. } t \in [0, T].$$

The abnormal case, where $\lambda_0 = 0$, is the same as the one shown to be impossible in part (b), so we may proceed with $\lambda_0 = 1$. For each t, the value $u(t)$ must maximize the function below over $v \in [-1, 1]$:

$$v \mapsto p_2(t)v - |v| = \begin{cases} (p_2(t) - 1)v, & \text{if } v \geq 0, \\ (p_2(t) + 1)v, & \text{if } v < 0. \end{cases}$$

This function is piecewise linear, so its maximizing point in $[-1, 1]$ will be one of -1, 0, or $+1$, depending on the slopes of its two segments:

$$\begin{aligned} p_2(t) > 1 \quad &\Rightarrow \quad p_2(t) + 1 > p_2(t) - 1 > 0 && \Rightarrow \quad u(t) = 1, \\ -1 < p_2(t) < 1 \quad &\Rightarrow \quad p_2(t) + 1 > 0 > p_2(t) - 1 && \Rightarrow \quad u(t) = 0, \\ p_2(t) < -1 \quad &\Rightarrow \quad p_2(t) - 1 < p_2(t) + 1 \leq 0 && \Rightarrow \quad u(t) = -1. \end{aligned}$$

Since $p_2(t) = mt + b$ is monotonic, the same will be true of $u(\cdot)$, and we need $x_2(\cdot)$ to have some positive values to satisfy the final constraint on $x_1(\cdot)$. So the only possible extremal control structure is, for some subinterval $[r, s]$ of $[0, T]$,

$$u(t) = \begin{cases} 1, & \text{for } 0 < t < r, \\ 0, & \text{for } r < t < s, \\ -1, & \text{for } s < t < T. \end{cases}$$

The switching times r and s are determined by

$$1 = p_2(r) = mr + b, \qquad -1 = p_2(s) = ms + b. \tag{$*$}$$

Integration gives

$$x_2(t) = \begin{cases} t, & \text{for } 0 \leq t \leq r, \\ r, & \text{for } r \leq t \leq s, \\ r + s - t, & \text{for } s \leq t \leq T. \end{cases}$$

The condition $x_2(T) = 0$ gives $s = T - r$. Now for $0 < t < r$,

$$0 = p_1(t)x_2(t) + p_2(t)u(t) - 1 - |u(t))| = -mt + (mt + b) - 1 - 1 = b - 2.$$

Hence $b = 2$ and $(*)$ yields $mr = -1$ and $m(T - r) = -3$, so $m = -1/r$ and $T - r = 3r$, i.e., $r = T/4$. Hence $s = 3T/4$ and we have

$$\begin{aligned} 0 \leq t \leq \frac{T}{4} \quad &\Rightarrow \quad x_2(t) = t, && x_1(t) = \tfrac{1}{2}t^2, \\ \frac{T}{4} \leq t \leq \frac{3T}{4} \quad &\Rightarrow \quad x_2(t) = \frac{T}{4}, && x_1(t) = \frac{T}{4}t - \frac{T^2}{32}, \\ \frac{3T}{4} \leq t \leq T \quad &\Rightarrow \quad x_2(t) = T - t, && x_1(t) = Tt - \tfrac{1}{2}t^2 - \frac{5}{16}T^2. \end{aligned}$$

The endpoint condition gives

$$D = x_1(T) = \frac{3}{16}T^2, \quad \text{so} \quad T = 4\sqrt{\frac{D}{3}}.$$

In the special case $D = 69$, we get $T = 4\sqrt{23} \approx 19.1833$.

3. We use the notation from the class notes and online writeup. Given $M(x) = -2\delta x$ and $N(x) = x$, we calculate

$$\Delta(x) = M'(x) + \delta N(x) = -2\delta + \delta x = \delta(x-2).$$

The unique zero of $\Delta()$ is $x^* = 2$.

The switching function ψ obeys the alternate adjoint equation

$$\dot{\psi} = (\delta - f_x(x,\widehat{u}))\psi + \Delta(x) = (\delta + 1 - \widehat{u}(t))\psi + \delta(x(t) - 2). \qquad (AAE)$$

An Unlikely Situation. We know $x(0) = \xi = 1 < x^*$, so if $\psi(0) < 0$ the only extremal evolution obeys $\widehat{u}(t) = 0$ and $\dot{x} = A(x) = -x$ for all t. Therefore $x(t) = e^{-t}$ for all t, and the equation for ψ says

$$\dot{\psi} = (\delta + 1)\psi + \delta e^{-t} - 2\delta.$$

The general solution is

$$\psi(t) = Ke^{(\delta+1)t} - \left(\frac{\delta}{2+\delta}\right)e^{-t} + \frac{2\delta}{\delta+1}.$$

Transversality requires

$$\begin{aligned} \psi(T) &= -e^{\delta T}\ell'(x(T)) - N(x(T)) \\ Ke^{(\delta+1)T} - \left(\frac{\delta}{2+\delta}\right)e^{-T} + \frac{2\delta}{\delta+1} &= -ke^{\delta T} - e^{-T}, \end{aligned}$$

We can solve for K,

$$K = -e^{-(\delta+1)T}\left[\frac{2\delta}{1+\delta} + \left(\frac{2}{2+\delta}\right)e^{-T} + ke^{\delta T}\right],$$

and use the result to investigate the initial value for ψ:

$$\begin{aligned} \psi(0) &= K - \left(\frac{\delta}{2+\delta}\right) + \frac{2\delta}{\delta+1} \\ &= -e^{-(\delta+1)T}\left[\frac{2\delta}{1+\delta} + \left(\frac{2}{2+\delta}\right)e^{-T} + ke^{\delta T}\right] - \left(\frac{\delta}{2+\delta}\right) + \frac{2\delta}{\delta+1} \\ &= \frac{\delta(3+\delta)}{(2+\delta)(1+\delta)} - ke^{-T} - e^{-(1+\delta)T}\left(\frac{2\delta}{1+\delta}\right) - e^{-(2+\delta)T}\left(\frac{2}{2+\delta}\right). \end{aligned}$$

When $\delta T \gg 1$ the last two terms will be negligible compared to the others, and our condition that $\psi(0) < 0$ becomes, approximately,

$$k \geq \frac{\delta(3+\delta)}{(2+\delta)(1+\delta)}e^T.$$

That is, driving the state down as aggressively as possible for the entire planning interval is optimal only in cases where a truly enormous penalty is assigned to the final value $x(T)$.

Initial Increase. It's more likely that $\psi(0) > 0$ so that $\dot{x}(t) = B(x(t)) = 1$ on some initial interval, so $x(t) = 1+t$ there. This puts the state on track to hit the level $x^* = 2$ at time $a = 1$. A premature sign change in ψ may force a switch to a permanent strategy of $\dot{x} = A(x) = -x$ before time a, but (much as before) this will require an enormous value of the cost coefficient k. Let's ignore this unlikely scenario.

So the state hits the level $x^* = 2$ at time $a = 1$. If it happens that $\psi(a) > 0$, then the state trajectory will follow $x(t) = 1 + t$ for the entire interval $[0, T]$, and the transversality condition will reveal that this happens if and only if $k \ll -1$. Let's assume k has a reasonable intermediate value, which leaves $\psi(a) = 0$ as the only option.

Now we know the first two segments of the extremal state trajectory (recall $x^* = 2$):

$$x(t) = \begin{cases} 1+t, & \text{for } 0 \le t \le 1, \\ 2, & \text{for } 1 \le t \le b. \end{cases}$$

It remains to identify the final interval and to decide what the state does there.

Final State Preview. Since $\ell'(x) = k$ is independent of x and $N(x) = x$, the transversality condition says

$$\psi(T) = -ke^{\delta T} - x(T).$$

The only way for $b = T$ and $x(T) = x^*$ to be extremal is for

$$0 = \psi(T) = -ke^{\delta T} - x^*, \qquad \text{i.e.,} \qquad k = -x^* e^{-\delta T} =: k^*.$$

Rearranging $x^* = -k^* e^{\delta T}$ lets us express the transversality condition as

$$\psi(T) + (x(T) - x^*) = -(k - k^*)e^{\delta T}.$$

Both terms on the left side have the same sign, so we will have $x(T) > x^*$ if $k < k^*$ and $x(T) < x^*$ if $k > k^*$. This makes some sense: if the penalty rate k attached to the final state $x(T)$ in our optimization goal is big, then we should act to decrease the final state near the final time.

Final Scenario 1–State Increase. Suppose $k < k^*$. Then $\dot{x} = B(x) = 1$ on the final interval $[b, T]$, so $x(t) = 2 + (t - b)$ there. This corresponds to $\widehat{u} = +1$, and (AAE) says

$$\dot{\psi} = (\delta + 1 - \widehat{u}(t))\psi + \delta(x(t) - 2) = \delta\psi + \delta(t - b).$$

The general solution is

$$\psi(t) = Ke^{\delta t} - t + b - \frac{1}{\delta}, \qquad K \in \mathbb{R}.$$

The condition $\psi(b) = 0$ reveals $K = e^{-\delta b}/\delta$, so that

$$\psi(T) = \frac{e^{\delta(T-b)}}{\delta} - (T - b) - \frac{1}{\delta}.$$

Transversality requires

$$\psi(T) = -e^{\delta T}\ell'(x(T)) - N(x(T)) = -ke^{\delta T} - 2 - (T - b),$$

so we have a nonlinear equation in which the only unknown is the final switching time b:

$$\frac{e^{\delta(T-b)}}{\delta} - (T - b) - \frac{1}{\delta} = -ke^{\delta T} - 2 - (T - b).$$

Rearrangement leads to

$$e^{\delta(T-b)} = -k\delta e^{\delta T} + 1 - 2\delta, \quad \text{so} \quad T - b = \frac{1}{\delta}\log\left(1 - \delta\left[2 + ke^{\delta T}\right]\right).$$

Note that the choice $k = k^*$ recaptures $b = T$, and the right side returns positive values for $(T - b)$ when $k < k^*$, which is consistent with our expectations.

Final Scenario 2–State Decrease. Suppose $k > k^*$. This requires $\psi(T) < 0$, so $\dot{x} = A(x) = -x$ on the final interval $[b, T]$. Hence $x(t) = 2e^{-(t-b)}$ there. This corresponds to $\widehat{u} = 0$, and (AAE) says

$$\dot{\psi} = (\delta + 1)\psi + 2\delta(e^{b-t} - 1).$$

The general solution is

$$\psi(t) = Ke^{(1+\delta)t} - \frac{2\delta}{2+\delta}e^{b-t} + \frac{2\delta}{1+\delta}.$$

Using $\psi(b) = 0$ reveals $K = -e^{-(1+\delta)b}\left(\frac{2\delta}{(1+\delta)(2+\delta)}\right)$, and leads to

$$\psi(T) = \frac{2\delta}{1+\delta} - \left(\frac{2\delta}{(1+\delta)(2+\delta)}\right)e^{(1+\delta)(T-b)} - \left(\frac{2\delta}{2+\delta}\right)e^{-(T-b)}.$$

Transversality requires this to equal $-ke^{\delta T} - x(T)$, i.e.

$$\frac{2\delta}{1+\delta} - \left(\frac{2\delta}{(1+\delta)(2+\delta)}\right)e^{(1+\delta)(T-b)} - \left(\frac{2\delta}{2+\delta}\right)e^{-(T-b)} = -ke^{\delta T} - 2e^{-(T-b)}.$$

Let's write $\theta = T - b$ for the duration of the final interval. Then the equation above takes the form

$$0 = \frac{2\delta e^{(1+\delta)\theta}}{(1+\delta)(2+\delta)} - \frac{4e^{-\theta}}{2+\delta} - \frac{2\delta}{1+\delta} - ke^{\delta T},$$

or, better still, after introducing a common denominator,

$$0 = 2\delta e^{(1+\delta)\theta} - 4(1+\delta)e^{-\theta} - 2\delta(2+\delta) - k(1+\delta)(2+\delta)e^{\delta T}. \qquad (*)$$

To check this, note that $\theta = 0$ provides a solution exactly when

$$0 = -(1+\delta)(2+\delta)\left[2 + ke^{\delta T}\right], \qquad \text{i.e.,} \qquad k = k^*.$$

This is encouraging.

A Specific Case. When $T = 10$, $k = 1/2$, and $\delta = 1/10$, equation $(*)$ becomes

$$0 = \frac{1}{5}e^{11\theta/10} - \frac{22}{5}e^{-\theta} - \frac{21}{50} - \frac{231e^1}{200}.$$

According to Maple's `fsolve` command, this implies $\theta \approx 2.6906$, so the unique extremal state evolution is

$$x(t) = \begin{cases} 1+t, & 0 \le t \le 1, \\ 2, & 1 \le t \le b, \\ 2e^{b-t}, & b \le t \le 10, \end{cases} \qquad \text{where} \quad b = T - \theta \approx 7.3094.$$