Math 403 Problem Set 9

Due in class on Friday 30 November 2012

1. Consider these two curves in the (t, x)-plane:

$$S_0 = \left\{ (t,x) \,:\, 0 \le t \le \frac{\pi}{2},\ x = \sin(2t) \right\}, \qquad S_1 = \left\{ \left(\frac{\pi}{2}, x \right) \,:\, x \in \mathbb{R} \right\}.$$

Among all piecewise smooth functions $x\colon [a, b] \to \mathbb{R}$ for which $(a, x(a)) \in S_0$ and $(b, x(b)) \in S_1$, find the one that provides a true minimum for

$$\int_a^b \left(\frac{\dot{x}(t)^2}{2} - \frac{x(t)^2}{2} \right) dt.$$

Provide a good sketch of S_0, S_1, and the graph of the minimizing arc. (Suggestion: Combine Dynamic Programming and ad-hoc reasoning.)

2. The activities below are inspired by this free-final point problem in the Calculus of Variations:

$$\min_{x(\cdot)} \left\{ -\frac{e^6}{2} x(1)^2 + \int_0^1 e^{x(t)\dot{x}(t)}\, dt \,:\, x(0) = 2 \right\}. \qquad P(0,2)$$

(a) Consider this family of related problems, defined using constants $k > 0$, $\tau < 1$ and $\xi > 0$:

$$\begin{aligned} \text{minimize} \quad & -\frac{k}{2} x(1)^2 + \int_\tau^1 e^{x(t)u(t)}\, dt \\ \text{subject to} \quad & \dot{x}(t) = u(t) \quad \text{a.e. } t \in [\tau, 1], \\ & x(t) > 0 \quad \forall t \in [\tau, 1), \\ & x(\tau) = \xi. \end{aligned} \qquad P(\tau, \xi)$$

Provide a simple formula for the "value function" $V(\tau, \xi) = \min P(\tau, \xi)$.
[*Suggestion*: "Guess" $V(\tau, \xi)$ by conjecturing that extremals are actually minimizers, then verify that this conjecture is actually correct.]

(b) Identify a function $F = F(t, x)$ with the property that for every initial point (τ, ξ) with $\tau < 1$ and $\xi > 0$, the solution of

$$\dot{x}(t) = F(t, x(t)), \qquad x(\tau) = \xi$$

provides a true minimizer in the problem defining $V(\tau, \xi)$.

(c) Taking $k = e^6$, sketch various solutions to the IVP in (b) in the same set of (t, x)-axes; highlight the true minimizer in $P(0, 2)$ above.

3. Consider the following three-dimensional system involving a constant $\varepsilon \ge 0$:

$$\dot{x} = y(1 - z), \quad \dot{y} = -x(1 - z) - \varepsilon y, \quad \dot{z} = -z(1 - z).$$

(i) Prove that the origin is a stable equilibrium point for every $\varepsilon \ge 0$.

(ii) Prove that the origin is asymptotically stable whenever $\varepsilon > 0$.

(iii) Prove that the origin is **not** asymptotically stable when $\varepsilon = 0$. (Clue: Investigate trajectories starting in the (x, y)-plane.)

(iv) Describe the set of initial points $\xi \in \mathbb{R}^3$ for which $\mathbf{x}(t; \xi) \to \mathbf{0}$ as $t \to \infty$.
[Here $\mathbf{x} = (x, y, z)$ and $t \mapsto \mathbf{x}(t; \xi)$ is the system trajectory starting from $\mathbf{x}(0; \xi) = \xi$.]

4. Consider the second-order system

$$\ddot{x} + 2\dot{x} + x(1 - x^2) = 0.$$

Use an energy-like function V to find a region of asymptotic stability for the constant solution $x(t) = 0$. (Here a *region of asymptotic stability* means an open set Ω in $(x, \dot{x})$-space with $(0, 0) \in \Omega$, such that if $(x(0), \dot{x}(0)) \in \Omega$, then $(x(t), \dot{x}(t)) \in \Omega$ for all $t \geq 0$ and $(x(t), \dot{x}(t)) \to (0, 0)$ as $t \to \infty$.)

5. Choose constant a, m, n so that the function $V(x, y) = x^m + ay^n$ is decreasing along every nonconstant trajectory of the planar system

$$\dot{x} = -x + 2y^3 - 2y^4, \qquad \dot{y} = -x - y + xy.$$

Deduce that this system can have no periodic orbits.