M403(2012) Solutions—Problem Set 9

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1. We know from class that for $\tau \in (0, \pi/2)$, the minimum cost of travelling from (τ, ξ) to S_1 is

$$V(\tau, \xi) = -\frac{\xi^2}{2}\frac{\cos\tau}{\sin\tau}.$$

Therefore the cheapest path from a starting point $(t, \sin(2t))$ in S_0 has cost

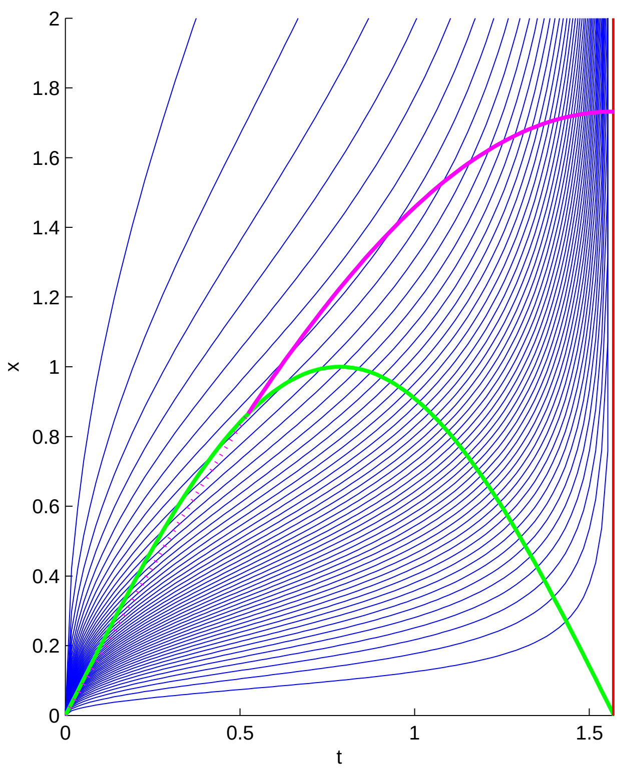
$$v(t) = V(t, \sin(2t)) = \frac{-\sin^2(2t)}{2}\frac{\cos t}{\sin t} = -2\sin t\cos^3 t.$$

To minimize this over the allowed interval, $0 \le t \le \pi/2$, we calculate

$$\begin{aligned} v'(t) &= 2\cos^4 t - 2\sin(t)(-3\cos^2 t\sin t) \\ &= -2\cos^2 t\left[\cos^2 t - 3\sin^2 t\right] \\ &= -2\cos^2 t\left[1 - 4\sin^2 t\right]. \end{aligned}$$

We have $v'(t) < 0$ for $0 < t < \pi/6$ and $v'(t) > 0$ for $\pi/6 < t < \pi/2$, so the choice $t = \pi/6$ gives a true minimum for $v(t)$ on the interval of interest. Therefore the cheapest way to get from S_0 to S_1 is to start from $(\pi/6, \sqrt{3}/2)$ and to follow the curve

$$x(t) = \sqrt{3}\,\sin(t), \qquad \frac{\pi}{6} \le t \le \frac{\pi}{2}.$$



The sketch above shows the launch set S_0 in green, the target set S_1 in red, and the optimum trajectory in magenta. The blue curves are level-curves of V, defined by

$$V(t, x) = c, \qquad c \in \mathbb{R}.$$

Notice that the optimal launch point is one where the level curve of V is tangent to the launch curve. This is compatible with the geometry of constrained optimization that leads to the Lagrange Multiplier approach.

2.

(a) In our general notation, this problem has

$$\ell(t,x) = -\frac{k}{2}x^2, \qquad f(t,x,u) = u, \qquad L(t,x,u) = e^{xu}, \qquad U = \mathbb{R}, \qquad S = \{(1,x) \,:\, x \in \mathbb{R}\}.$$

Therefore

$$\begin{aligned} H(t,x,p,u) &= pu - e^{xu}, \\ \widehat{U}(t,x,p) &= \arg\max_{u\in\mathbb{R}} H(t,x,p,u) = \left\{ \frac{1}{x}\log\left(\frac{p}{x}\right) \right\}, \\ \mathcal{H}(t,x,p) &= \frac{p}{x}\left(\log\left(\frac{p}{x}\right) - 1 \right) \end{aligned}$$

An extremal triple $(\widehat{T},\widehat{u},x)$ with starting point (τ,ξ) must satisfy $\dot{x} = \widehat{u}$ a.e., and have some costate arc p for which the maximum condition holds a.e.:

$$\dot{x}(t) = \widehat{u}(t) \in \widehat{U}(t,x(t),p(t)), \qquad \text{i.e.,} \qquad x(t)\dot{x}(t) = \log\left(\frac{p(t)}{x(t)}\right).$$

Also, the adjoint equation requires

$$-\dot{p}(t) = H_x(t,x(t),p(t),\widehat{u}(t)) = -\widehat{u}(t)e^{x(t)\widehat{u}(t)} = -\dot{x}(t)e^{x(t)\dot{x}(t)}.$$

Substituting from above gives

$$-\dot{p}(t) = -\dot{x}(t)\left(\frac{p(t)}{x(t)}\right), \qquad \text{i.e.,} \qquad \frac{\dot{p}(t)}{p(t)} = \frac{\dot{x}(t)}{x(t)}.$$

This says that the functions $\log|p(t)|$ and $\log|x(t)|$ have identical derivatives, so they must differ by a constant. It follows that some constant N obeys

$$p(t) = Nx(t).$$

In particular, the right-hand expression $\log(p(t)/x(t)) = \log N$ above is actually constant; for convenience, let's define $M = 2\log N$ instead and work with the identity

$$2x(t)\dot{x}(t) = M.$$

The left side is the derivative of $x(t)^2$, so integration (and $x(\tau) = \xi$) gives

$$x(t)^2 = M(t-\tau) + \xi^2. \tag{2}$$

The transversality condition at the final time requires

$$-p(1) = \ell_x(x(1)) = -kx(1), \qquad \text{i.e.,} \qquad -Nx(1) = -kx(1), \qquad \text{i.e.,} \qquad N = k.$$

Thus the unique extremal starting from (τ,ξ) obeys

$$x(t)^2 = 2(t-\tau)\log k + \xi^2, \qquad \text{so} \qquad 2x(t)\dot{x}(t) = 2\log k.$$

Recall $\widehat{u} = \dot{x}$, so the objective value for this extremal is

$$\begin{aligned} W(\tau,\xi) &= -\frac{k}{2}\left[2(1-\tau)\log k + \xi^2\right] + \int_\tau^1 e^{\log k}\,dt \\ &= -k(1-\tau)\log k - \frac{k}{2}\xi^2 + (1-\tau)k \\ &= k(1-\log k)(1-\tau) - \tfrac{1}{2}k\xi^2. \end{aligned}$$

In more comfortable notation, let's consider

$$W(t,x) = k(1-\log k)(1-t) - \tfrac{1}{2}kx^2.$$

We have $W(t,x) = \ell(t,x) = -\frac{1}{2}kx^2$ at each point $(t,x) \in S$. Also,

$$W_t(t,x) = -k(1-\log k), \qquad W_x(t,x) = -kx$$

show that

$$\mathcal{H}(t,x,-W_x(t,x)) = \frac{-W_x(t,x)}{x}\left(\log\left(\frac{-W_x(t,x)}{x}\right)-1\right) = \frac{kx}{x}\left(\log\left(\frac{kx}{x}\right)-1\right) = k(\log k - 1).$$

Therefore W satisfies both the Hamilton-Jacobi equation and the required boundary condition, and the trajectories above have the correct velocities to confirm that each of them is a true minimizer. Also, the minimum value is $V(t,x) = W(t,x)$.

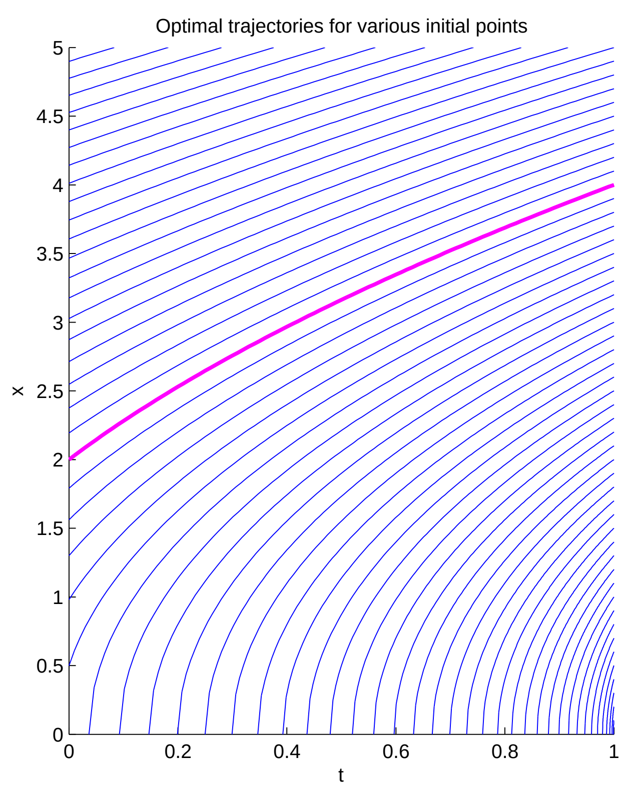
(b) The optimal feedback function is

$$F(t,x) = \widehat{U}(t,x,-V_x(t,x)) = \widehat{U}(t,x,kx) = \frac{\log k}{x}.$$

(c) When $k = e^6$, the feedback law becomes $\dot{x} = 6/x$. Rearrangement gives $2x\dot{x} = 12$, so the positive-valued solution through (τ,ξ) is

$$x(t) = \sqrt{\xi^2 + 12(t-\tau)}.$$

In particular, when $(\tau,\xi) = (0,2)$ the minimizer is $x(t) = \sqrt{4+12t}$; the optimal final point is $x(1) = 4$. Here is the required figure.



3. (i) Let $V(x,y,z) = x^2+y^2+z^2$ and $F(x,y,z) = (y(1-z),\, -x(1-z)-\varepsilon y,\, -z(1-z))$. Note that $V(x,y,z) > 0$ whenever $(x,y,z) \neq (0,0,0)$, and

$$\begin{aligned}\nabla V(x,y,z) \bullet F(x,y,z) &= (2x,\, 2y,\, 2z) \bullet (y(1-z),\, -x(1-z)-\varepsilon y,\, -z(1-z)) \\ &= -2\varepsilon y^2 - 2z^2(1-z).\end{aligned}$$

Let $G = \{(x,y,z) \,:\, z < 1\}$. Then $\nabla V \bullet F \leq 0$ on $D = G$, so the same inequality holds on each of the sub-level sets

$$\Omega_\ell = \left\{(x,y,z) \,:\, x^2+y^2+z^2 < \ell\right\}, \qquad 0 < \ell \leq 1.$$

Each of these sets is therefore flow-invariant. Given $\varepsilon > 0$, choose any $\ell \in (0, \varepsilon^2)$ to obtain an open flow-invariant set Ω_ℓ inside $\mathbb{B}[\mathbf{0};\varepsilon]$ containing $\mathbf{0}$. This confirms that $\mathbf{0}$ is a stable equilibrium for the given system.

(ii) Suppose $0 < \ell \leq 1$ and $\varepsilon > 0$. Consider

$$D_0 \stackrel{\text{def}}{=} \{(x,y,z) \in G \,:\, \nabla V(x,y,z) \bullet F(x,y,z) = 0\} = \{(x,0,0) \,:\, x \in \mathbb{R}\}.$$

An F-trajectory evolving entirely in D_0 must have $y \equiv 0$, $z \equiv 0$, and $\dot{y} \equiv 0$: these conditions imply $x \equiv 0$ from the second component-equation. So the only flow-invariant subset of D_0 is the one-point set $\{\mathbf{0}\}$, and it follows that Ω_ℓ is a domain of attraction for the equilibrium point at $\mathbf{0}$. Since a set Ω_ℓ can be squeezed inside any given $\mathbf{0}$-neighbourhood, $\mathbf{0}$ is an asymptotically stable equilibrium point.

(iii) The situation is quite different when $\varepsilon = 0$. For any $\alpha > 0$, the F-trajectory given by

$$(x(t), y(t), z(t)) = \alpha(\sin t,\, \cos t,\, 0)$$

starts within distance α of 0 and never gets any closer: since $\alpha > 0$ can be arbitrarily small, the origin fails to attract some trajectories that start nearby. It is not asymptotically stable.

(iv) Let $W(x,y,z) = x^2 + y^2$. Along F-trajectories,

$$\frac{dW}{dt} = \nabla W \bullet F = 2xy(1-z) - 2xy(1-z) - 2\varepsilon y^2 = -2\varepsilon y^2.$$

If $\varepsilon = 0$ this is 0, showing that W is constant along trajectories. The only initial points for which trajectories can converge to $\mathbf{0}$ must make $W = 0$, i.e., have $x \equiv 0$ and $y \equiv 0$. The domain of attraction for $\mathbf{0}$ is just the half-line $\{(0,0,z) \,:\, z < 1\}$.

If $\varepsilon > 0$, using $V(x,y,z) = x^2 + y^2 + m^2 z^2$ in the arguments of (ii) will show that every ellipsoid of the form below is a domain of attraction for the origin:

$$x^2 + y^2 + m^2 z^2 < m^2, \qquad m > 0.$$

For any initial point (x_0, y_0, z_0) with $|z_0| < 1$, a suitable choice of m will show that the trajectory converges to $\mathbf{0}$. For any initial point with $z_0 \leq -1$, any interval $[0, \theta]$ on which $z(t) \leq -1$ will have

$$\dot{z}(t) = -z(1-z) \geq 1 - z \geq 2,$$

so the trajectory will enter the region where $z > -1$ in finite time; after that, it will converge to $\mathbf{0}$ by the argument above. So the domain of attraction for $\mathbf{0}$ is the half-space defined by $z < 1$.

4. With $x_1 = x$, $x_2 = \dot{x}$, the system becomes

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1(1 - x_1^2) - 2x_2.\end{aligned}$$

Motivated by analogy with a physical system, we consider the energy-like function

$$V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} x(1 - x^2)\,dx = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4.$$

For this function, we have

$$\begin{aligned}\nabla V(x) \bullet F(x) &= \left(x_1 - x_1^3, x_2\right) \bullet \left(x_2,\, -x_1(1 - x_1^2) - 2x_2\right)\\ &= x_1x_2 - x_1^3x_2 - x_1x_2 + x_1^3x_2 - 2x_2^2 = -2x_2^2.\end{aligned}$$

The result is never positive, so—in the notation of Recipe E.8—we have $D = \mathbb{R}^2$. Hence any bounded component of any level set $\Omega_\ell = \left\{x \in \mathbb{R}^2 \,:\, V(x) < \ell\right\}$, $\ell \in \mathbb{R}$, is flow-invariant. Here

$$\begin{aligned}x \in \Omega_\ell \quad &\Leftrightarrow \quad \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4 < \ell\\ &\Leftrightarrow \quad \frac{1}{2}x_2^2 - \frac{1}{4}\left(x_1^2 - 1\right)^2 < \ell - \frac{1}{4},\end{aligned}$$

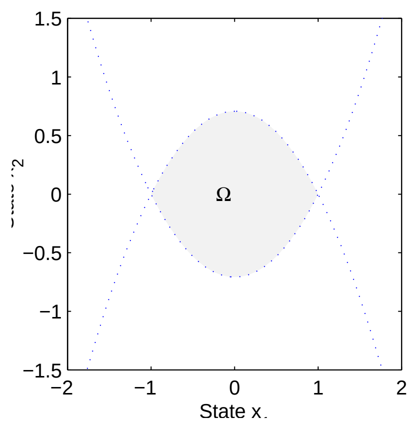
so Ω_ℓ is an open set symmetric with respect to both the coordinate axes. When $\ell = 1/4$, its boundary is formed by the two parabolas $x_2 = \pm\dfrac{1}{\sqrt{2}}\left(x_1^2 - 1\right)$. The bounded part of $\Omega_{1/4}$ may be chosen for Ω. With this choice, we have

$$D_0 = \{x \in \Omega \,:\, \nabla V(x) \bullet F(x) = 0\} = \left\{x \in \mathbb{R}^2 \,:\, -1 < x_1 < 1,\ x_2 = 0\right\}.$$

Any function $x(t)$ which obeys $\dot{x} = F(x)$ while lying inside D_0 must obey $x_2(t) \equiv 0$. This implies both $\dot{x}_1(t) \equiv 0$ and $\dot{x}_2(t) \equiv 0$, and forces $x_1(t)$ to be a constant function satisfying

$$x_1\left(1 - x_1^2\right) = 0.$$

This nonlinear equation has three solutions: $x_1 = -1$, $x_1 = 0$, and $x_1 = +1$. (This is consistent with the observation that the system has three equilibrium points, $(-1, 0)$, $(0, 0)$, and $(+1, 0)$.) However, only the solution $x_1 = 0$ corresponds to a point in D_0, so every solution starting in Ω converges to 0 as $t \to \infty$.



5. Along any trajectory of the given system, we have

$$\begin{aligned}\frac{d}{dt}V(x,y) &= mx^{m-1}\dot{x} + any^{n-1}\dot{y} \\ &= mx^{m-1}\left[-x+2y^3-2y^4\right] + any^{n-1}\left[-x-y+xy\right] \\ &= -mx^m + 2mx^{m-1}y^3 - 2mx^{m-1}y^4 - anxy^{n-1} - any^n + anxy^n.\end{aligned}$$

The exponents of y appearing in this expression are $3, 4, n-1, n$, so it makes sense to try $n = 4$: this reduces the expression above to

$$\begin{aligned}\frac{d}{dt}V(x,y) &= -mx^m + 2mx^{m-1}y^3 - 2mx^{m-1}y^4 - 4axy^3 - 4ay^4 + 4axy^4 \\ &= -mx^m + y^3\left[2mx^{m-1} - 4ax\right] - y^4\left[4a + 2mx^{m-1} - 4ax\right].\end{aligned}$$

Now to put a zero coefficient next to the y^3 term, we need first to match the exponents on x by choosing $m = 2$, and then to arrange $2m = 4a$, i.e., $a = m/2 = 1$. With these choices, we have $V(x,y) = x^2 + y^4$ and

$$\frac{d}{dt}V(x,y) = -2x^2 - y^3\,[0] - y^4\,[4 + 4x - 4x] = -2x^2 - y^4 \le 0.$$

Thus V is nonincreasing along every trajectory. In fact, one has $dV/dt < 0$ at every point except $(0,0)$, and the only system trajectory containing this point is the constant equilibrium trajectory $(x(t), y(t)) = (0,0)$. Hence, as stated, every nonconstant trajectory is one along which V decreases strictly. This makes a periodic orbit impossible, because along a periodic orbit the V-values would exhibit periodic behaviour too, and this is incompatible with the strict decrease property just proved.